Moment Integrations for 3D Printed Shapes

Nobuyuki Umetani
Autodesk Research

AUTODESK

[Prevost et al. 2013]

Overview

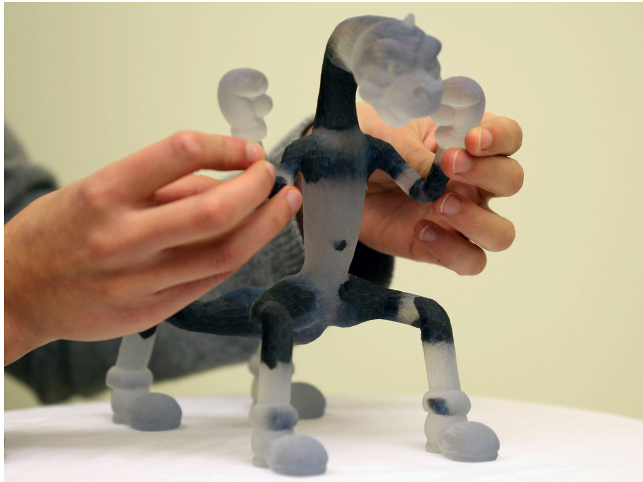
- Intro
- Integration over 3D domain
- Mass
- Center of Mass
- Moment of Inertia

Inhomogeneous Material Distribution

color

http://www.stratasys.com/

stiffness



[Skouras et al.2013]

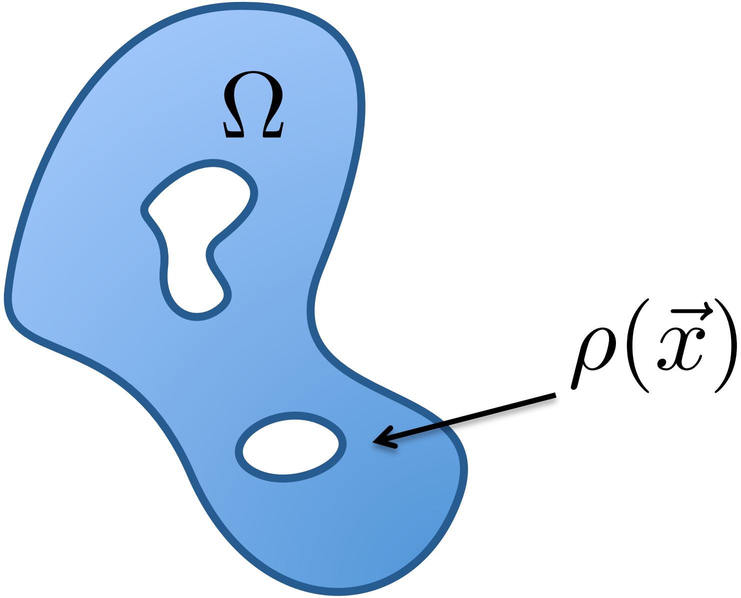
Inhomogeneous Density Distribution

Built-to-last, [Lu et al. 2014]

Buoyancy Optimization, [Wang et al. 2016]

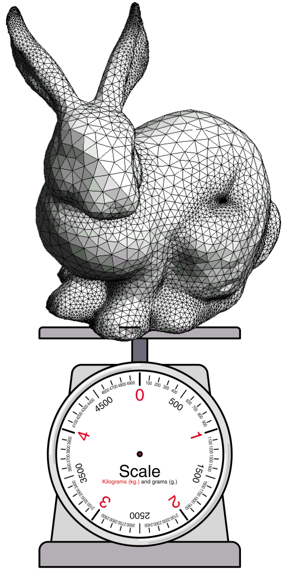
Measure of 3D Density Distribution

- How much density information can you still measure without seeing 'inside' of a 3D shape?

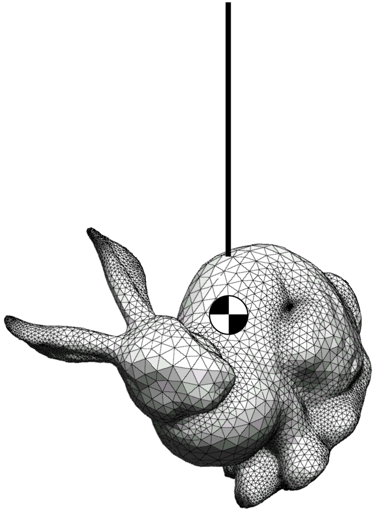


Invariants of 3D Density Distribution

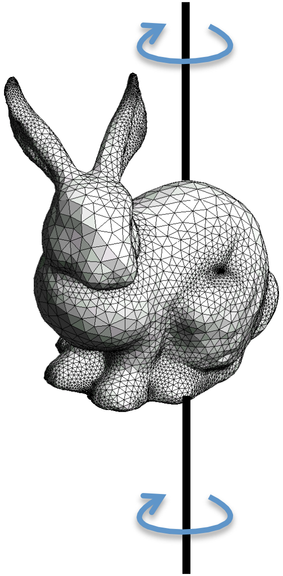
Put on the scale



Hang on a string

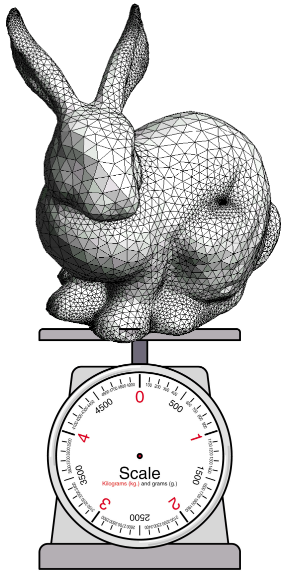


Rotate on an axis

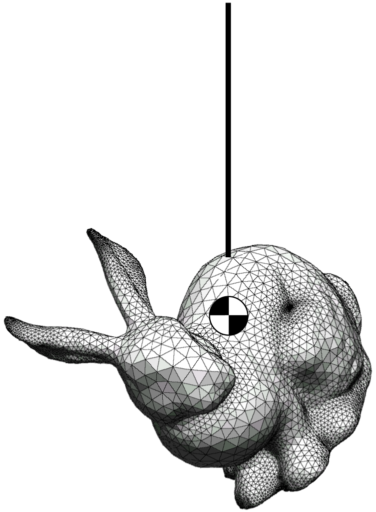


Invariants of 3D Density Distribution

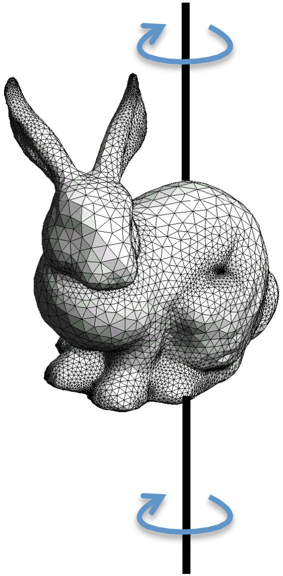
Mass



Center of Mass



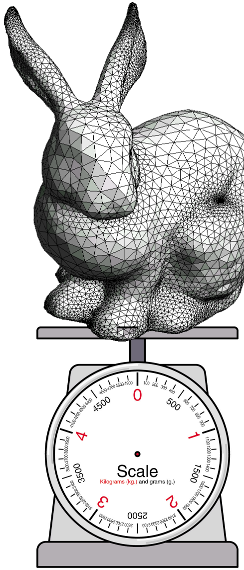
Moment of Inertia

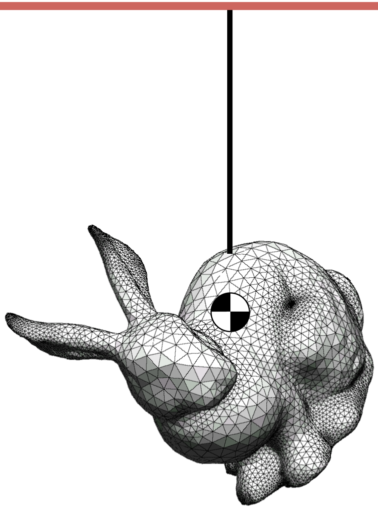


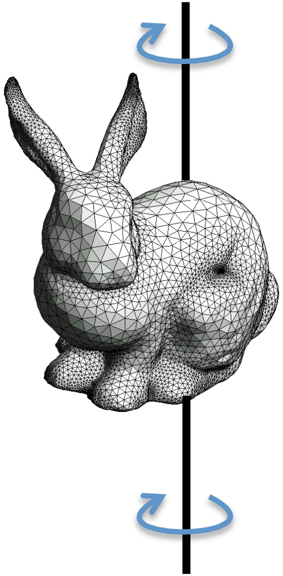
Invariants of 3D Density Distribution

Mass	Center of Mass	Moment of Inertia
$\int_\Omega (\vec{x})^0 dv$	$\int_\Omega (\vec{x})^1 dv$	$\int_\Omega (\vec{x})^2 dv$

Integration characterize these properties







Overview

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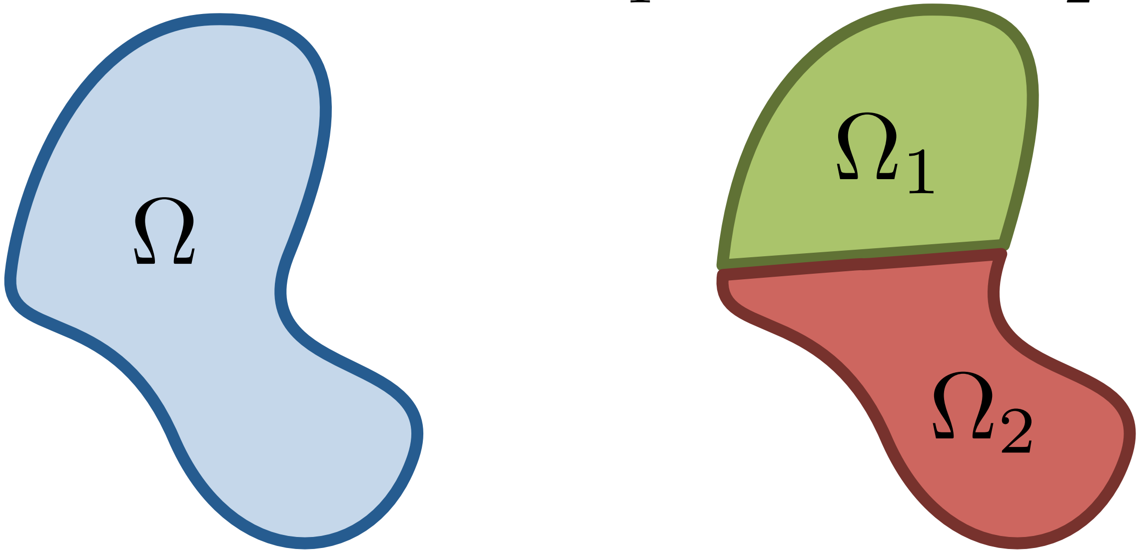
Integration Formula is Convenient

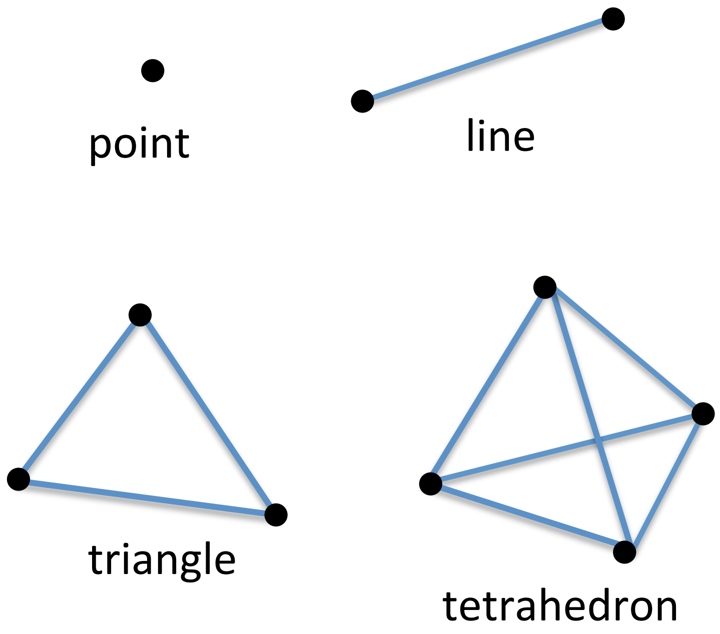
Decomposing the integration region into simpler shapes

$$\int_{\Omega} f\,dv = \int_{\Omega_1} f\,dv + \int_{\Omega_2} f\,dv$$



Integration Rules for the Simplex Shape

- Simplex shape



Integration Rules for the Simplex Shape

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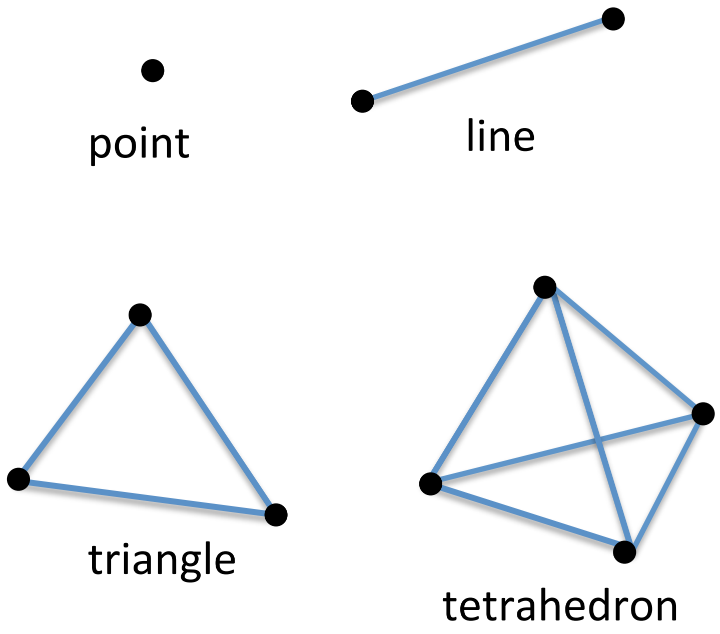
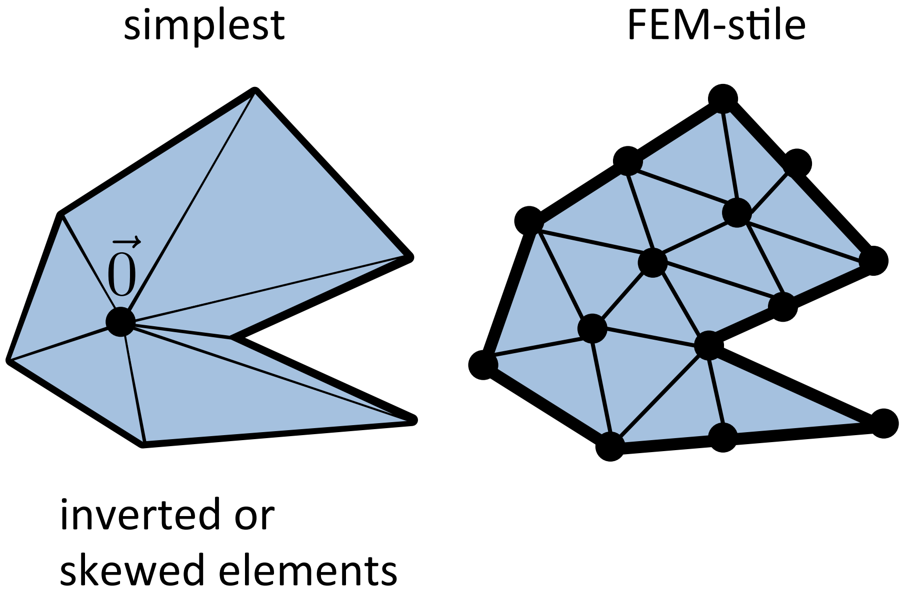
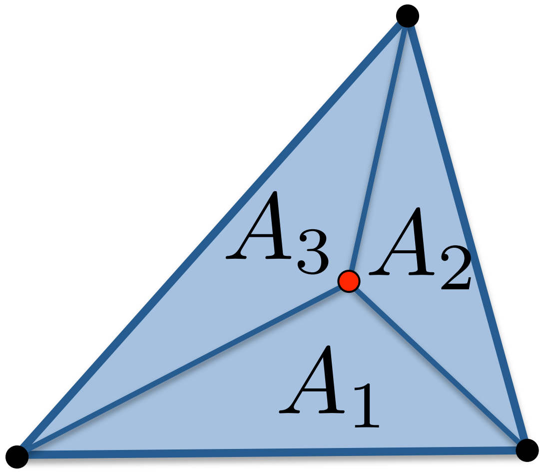


- Decomposition



Integration Rules: Analytical

Barycentric coordinate (L_1, L_2, L_3)

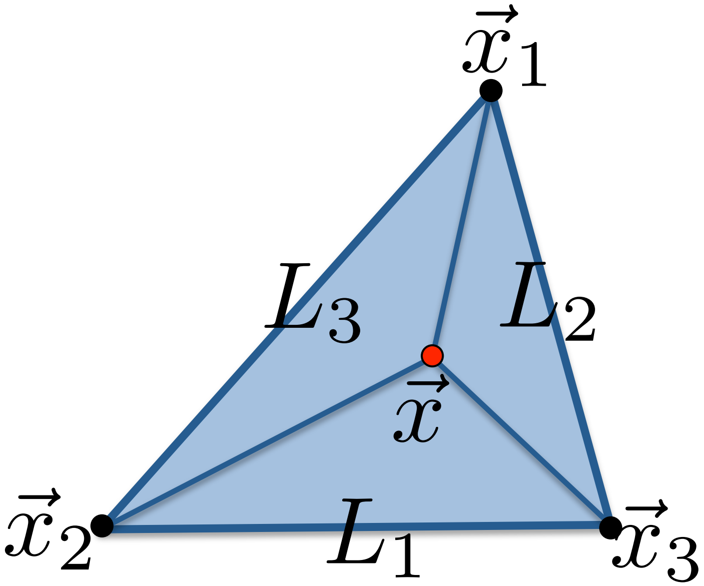


$$A = A_1 + A_2 + A_3$$

$$L_i = A_i / A$$

Integration Rules: Analytical

Position and integrand are both represented using the Barycentric coordinate



position

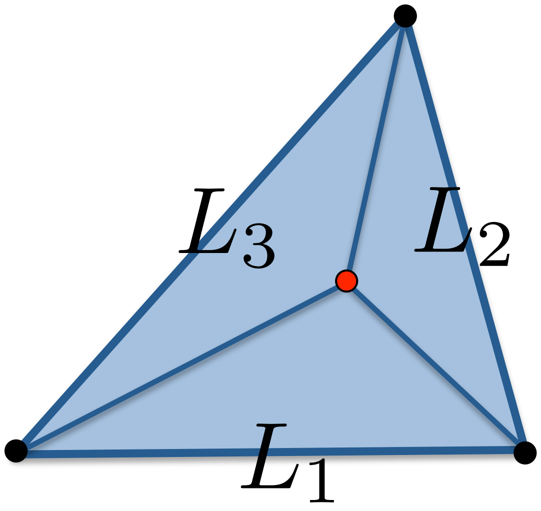
$$\vec{x} = \vec{x}_1 L_1 + \vec{x}_2 L_2 + \vec{x}_3 L_3$$

integrand

$$f(\vec{x}) = f(L_1, L_2, L_3)$$

Integration Rules: Analytical

For polynomial integrands, there is an analytic integration formula



$$f(L_1, L_2, L_3) = L_1^a L_2^b L_3^c$$

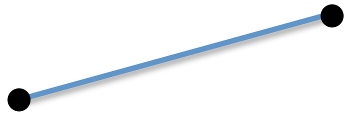
Integration over the triangle

$$\int_A L_1^a L_2^b L_3^c \, da = \frac{a!b!c!2!}{(a+b+c+2)!} A$$

Integration Rules: Analytical

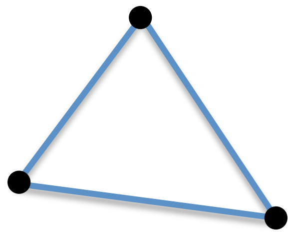
1D

$$\int_L L_1^a L_2^b dl = \frac{a!b!1!}{(a+b+1)!} L$$



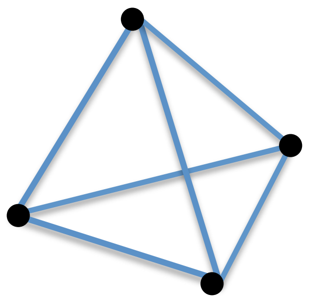
2D

$$\int_A L_1^a L_2^b L_3^c da = \frac{a!b!c!2!}{(a+b+c+2)!} A$$



3D

$$\int_V L_1^a L_2^b L_3^c L_4^d dv = \frac{a!b!c!d!3!}{(a+b+c+d+3)!} V$$

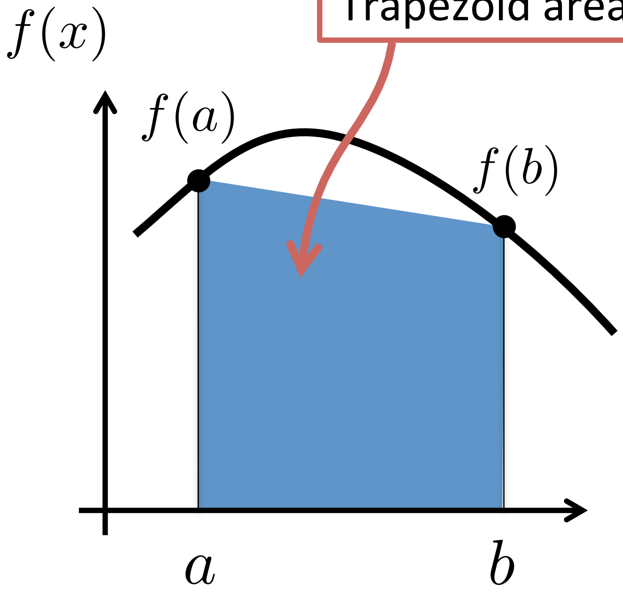


Proof of them is VERY difficult (need the Gamma function) https://en.wikipedia.org/wiki/Beta_function

Integration Rules: Numerical

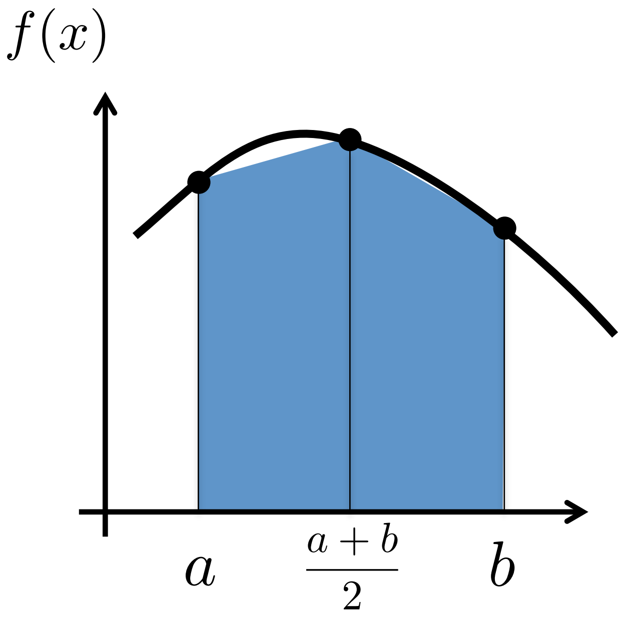
Trapezoid rule

Trapezoid area approximate integration

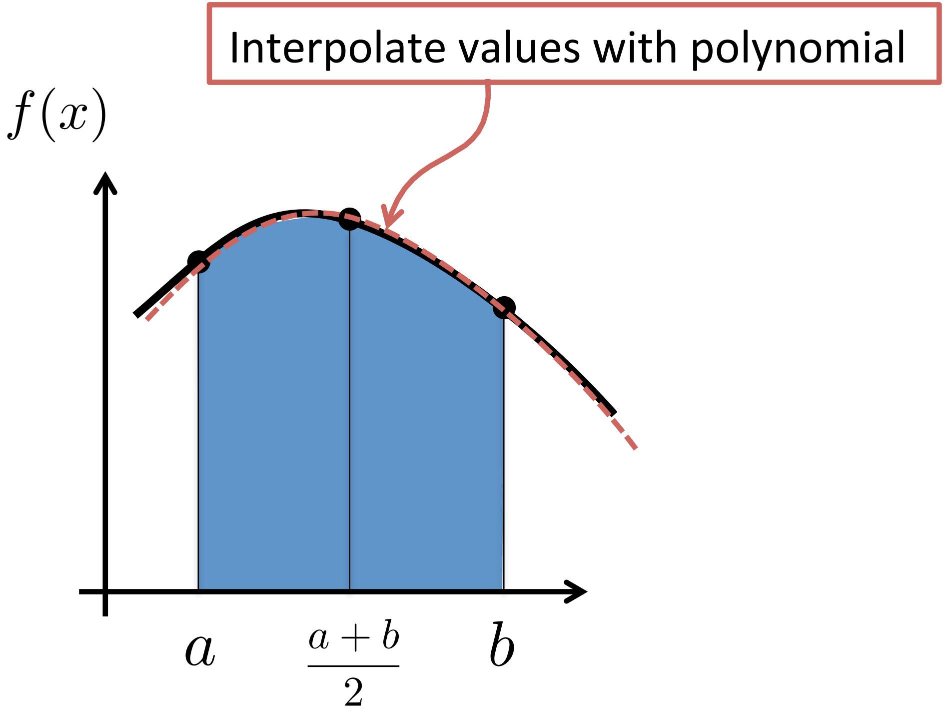


Integration Rules: Numerical

Trapezoid rule

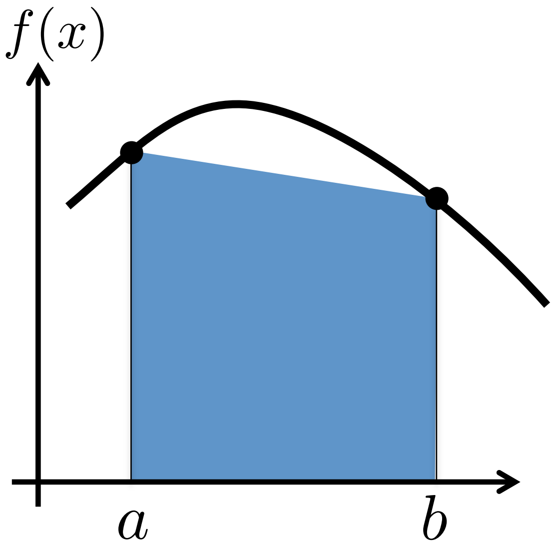


Newton-Cotes quadrature

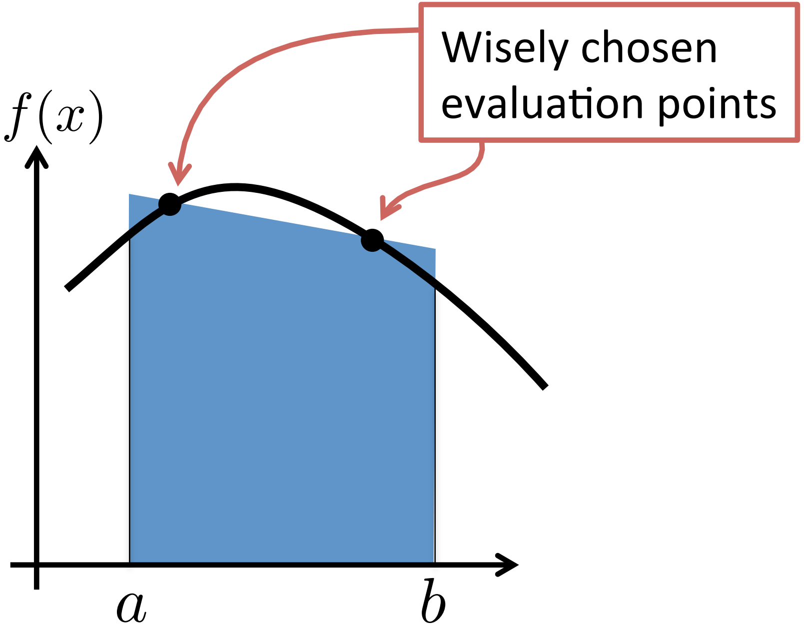


Integration Rules: Numerical

Trapezoid rule



Gauss quadrature



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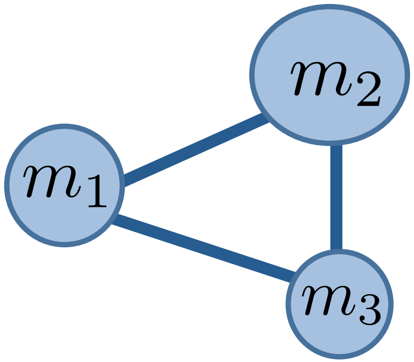
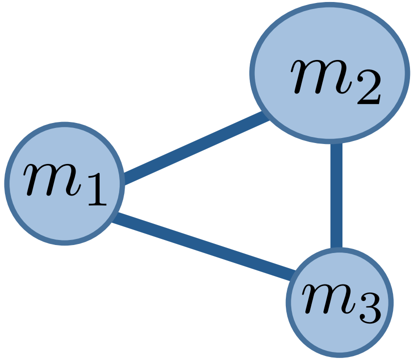
Mass

$$M = \sum m_i$$



Mass

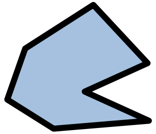
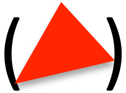
$$M = \sum m_i$$



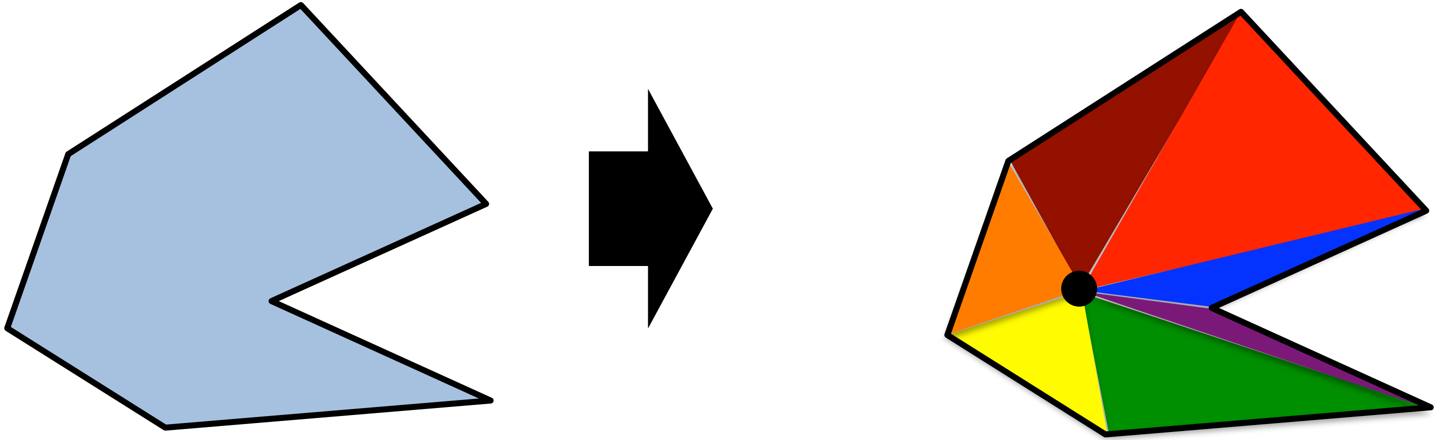
$$M = \int_V \rho(\vec{x})\,dV \xrightarrow{\text{Homogeneous density}} M = \rho \int_V dV$$

Volume of a Triangle Mesh Ⅰ

Computing volume integral using volume discretization

Area() = area()+area()+area()+…

Volume integral Divide the region into simplexes



Volume of a Triangle Mesh Ⅰ

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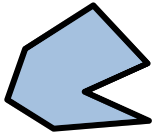
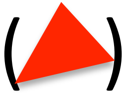
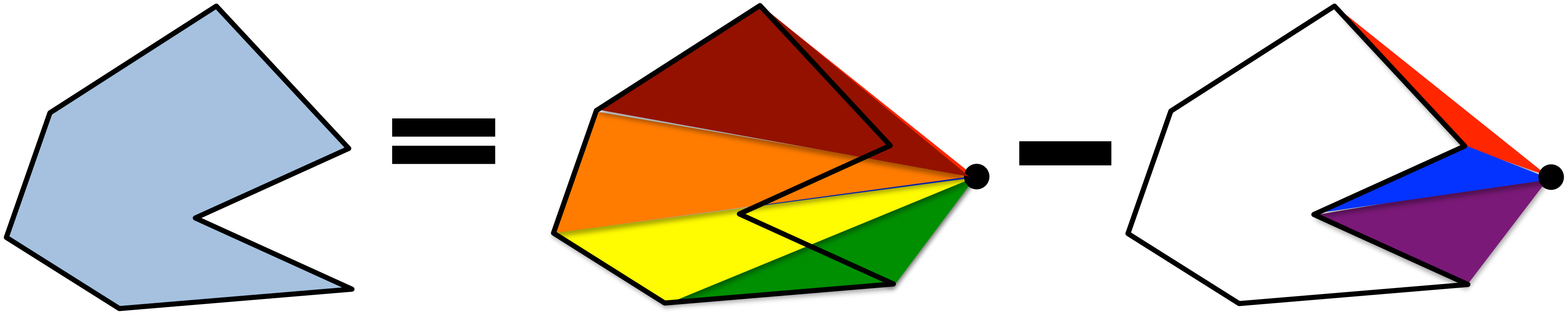
Area() = area()+area()+area()+…

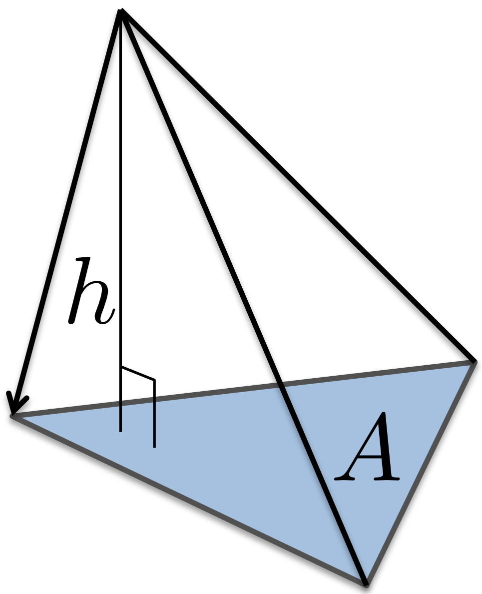
Volume integral

OK to be inverted. The area can be minus



Volume of a Triangle Mesh I

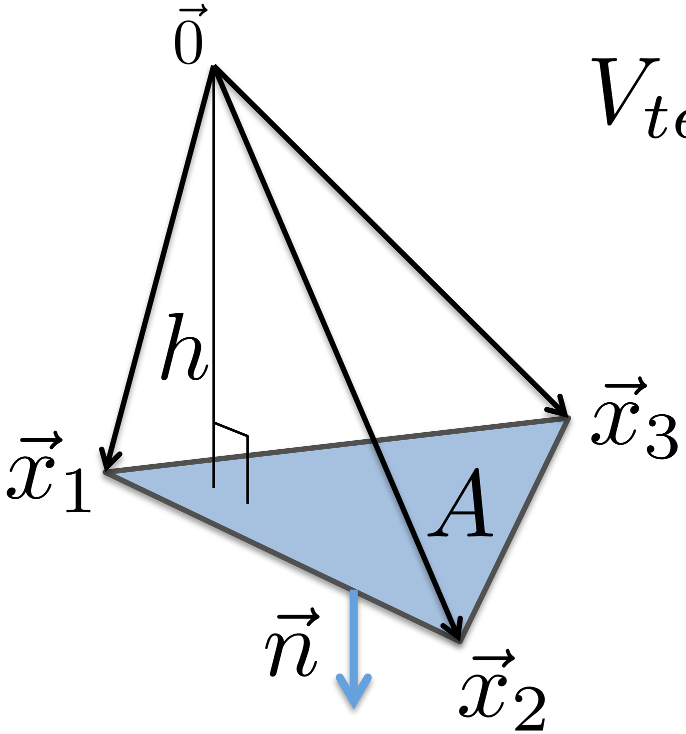
Volume of a tetrahedron



$$V_{tet} = \frac{1}{3}Ah$$

Volume of a Triangle Mesh I

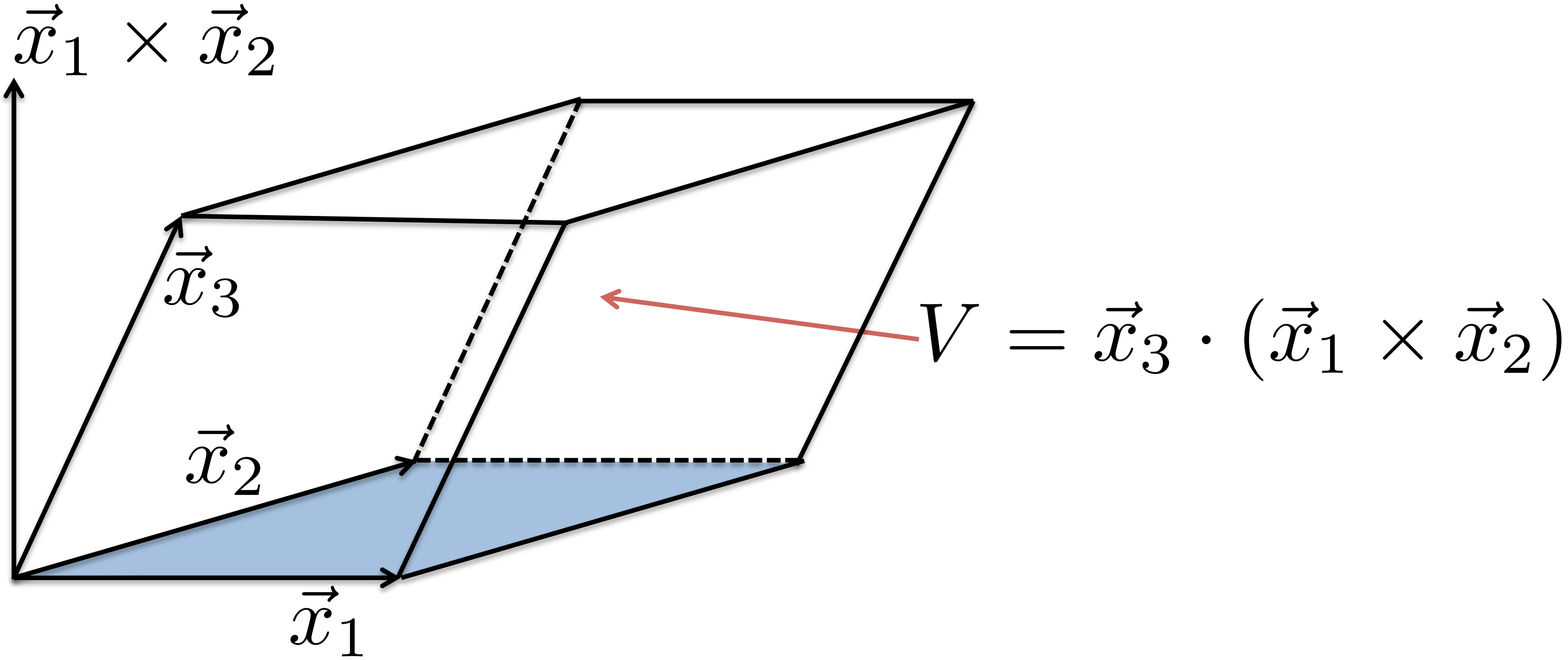
Volume of a tetrahedron



$$
\begin{aligned}
V_{tet} &= \frac{1}{3}Ah \\
&= \frac{1}{3}A(\vec{x}_1 \cdot \vec{n}) \\
&= \frac{1}{3}A\frac{\vec{x}_1 + \vec{x}_2 + \vec{x}_3}{3} \cdot \vec{n}
\end{aligned}
$$

Volume of a Triangle Mesh I

Volume of a tetrahedron can be computed from a parallelepiped



Volume of a Triangle Mesh I

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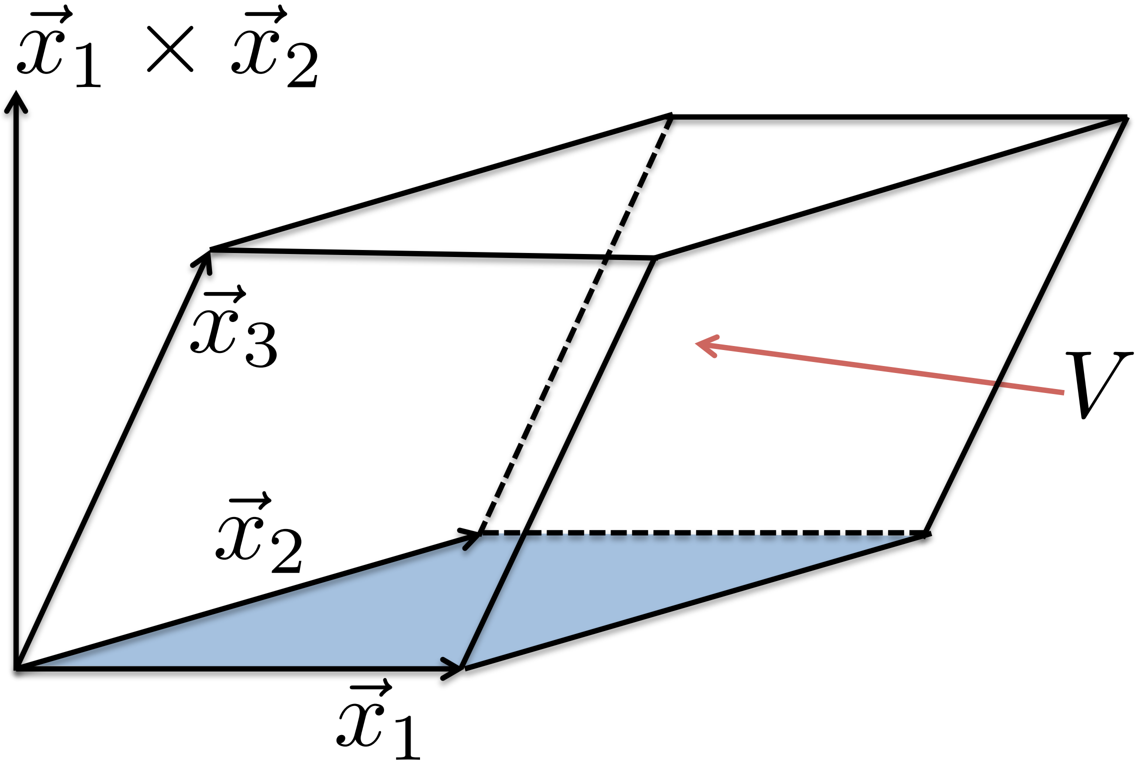
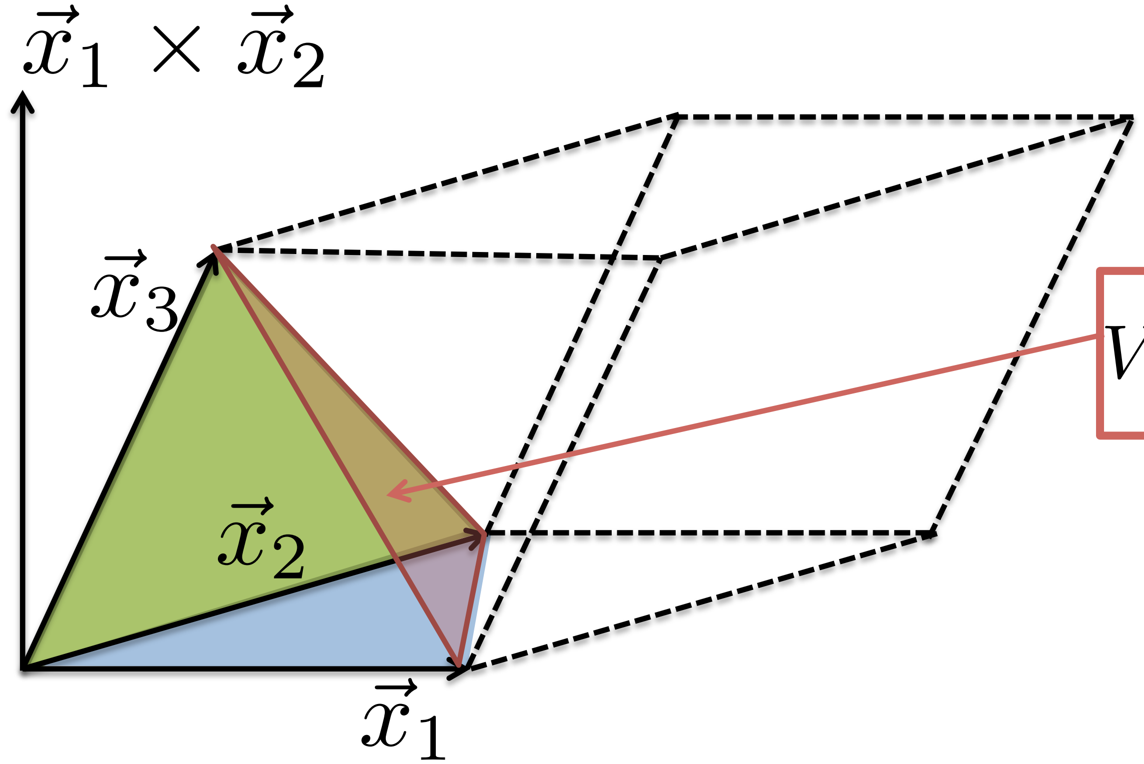


vector triple product

$$
\begin{aligned}
V &= \vec{x}_3 \cdot (\vec{x}_1 \times \vec{x}_2) \\
&= \vec{x}_1 \cdot (\vec{x}_2 \times \vec{x}_3) \\
&= \vec{x}_2 \cdot (\vec{x}_3 \times \vec{x}_1)
\end{aligned}
$$

Volume of a Triangle Mesh Ⅰ

Volume of a tetrahedron can be computed from a parallelepiped



$$V_{tet} = \frac{1}{6}\vec{x}_3 \cdot (\vec{x}_1 \times \vec{x}_2)$$

Volume of a Triangle Mesh Ⅱ

Computing volume integration using surface integration

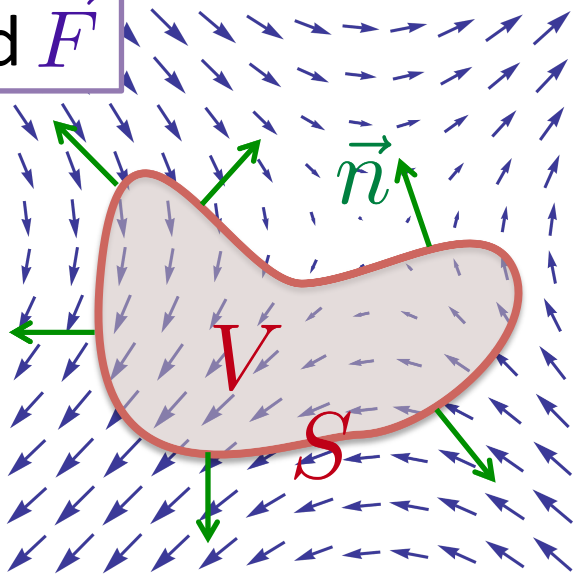
The Gauss's divergence theorem

$$\underbrace{\int_{\Omega} (\nabla \cdot \vec{F}) dv}_{\text{volume integration}} = \underbrace{\int_{\partial\Omega} (\vec{F} \cdot \vec{n}) ds}_{\text{surface integration}}$$

Volume of a Triangle Mesh Ⅱ

$$\boxed{\int_{\Omega} (\nabla \cdot \vec{F}) dv = \int_{\partial\Omega} (\vec{F} \cdot \vec{n}) ds}$$

vector field $\vec{F}$



measures total outward flow through V's boundary

Volume of a Triangle Mesh Ⅱ

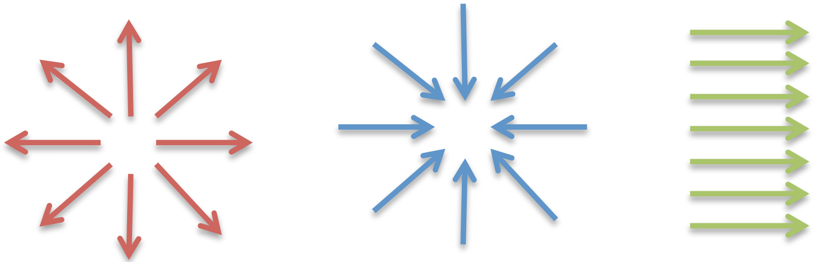
add up little bits of
outward flow in V

$$\boxed{\int_{\Omega}(\nabla \cdot \vec{F})dv = \int_{\partial\Omega}(\vec{F}\cdot\vec{n})ds}$$

divergence

$$\boxed{= \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}}$$

$\nabla \cdot \vec{F} > 0$ $\nabla \cdot \vec{F} < 0$ $\nabla \cdot \vec{F} = 0$



Volume of a Triangle Mesh Ⅱ

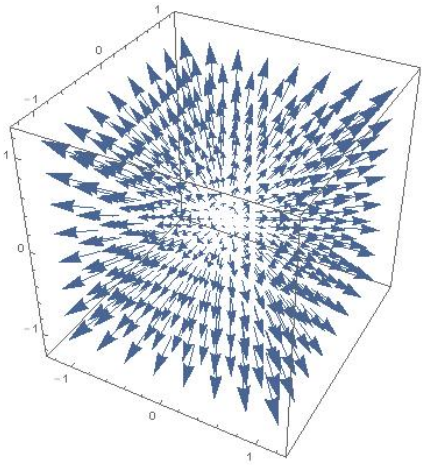
$$\boxed{\int_\Omega dv = \int_\Omega \underbrace{(\nabla \cdot \vec{F})}_{=1} dv = \int_{\partial\Omega} (\vec{F} \cdot \vec{n}) ds}$$

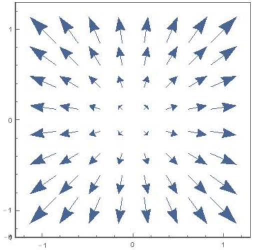
$\nabla \cdot \vec{F} = 1$ ➡ What's the $\vec{F}$ looks like?

Volume of a Triangle Mesh Ⅱ

One example for $\vec{F}$

$$\vec{F} = \frac{1}{3}\vec{x} = \frac{1}{3}(x\vec{e}_x + y\vec{e}_y + z\vec{e}_z)$$



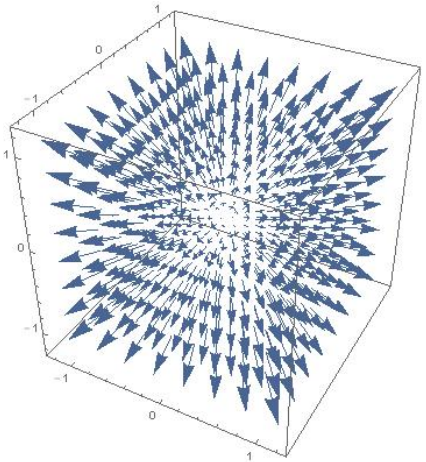


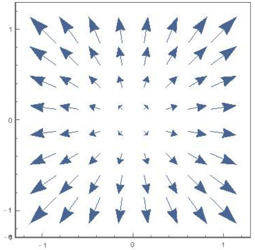
$$\int_\Omega dv = \int_{\partial\Omega} \left(\frac{1}{3}\vec{x}\cdot\vec{n}\right) ds = \sum_{t\in T_{ri}} \int_{\partial\Omega_t} \left(\frac{1}{3}\vec{x}\cdot\vec{n}\right) ds$$

Volume of a Triangle Mesh Ⅱ

One example for $\vec{F}$

$$\vec{F} = \frac{1}{3}\vec{x} = \frac{1}{3}(x\vec{e}_x + y\vec{e}_y + z\vec{e}_z)$$

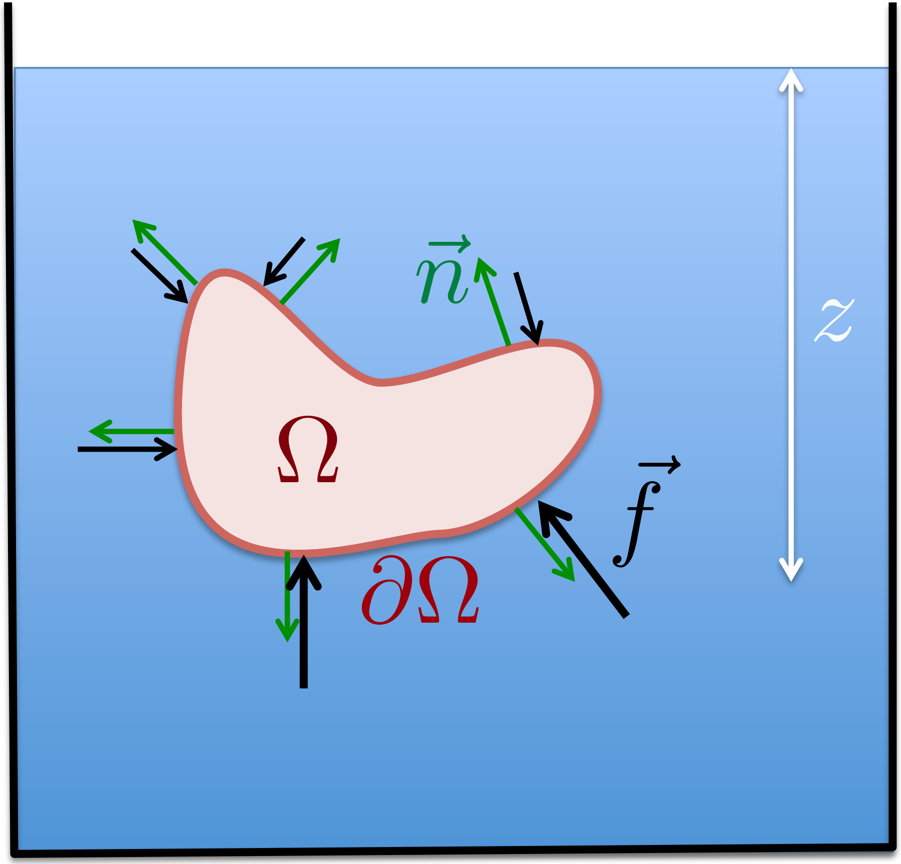




$$\int_{\Omega} dv = \int_{\partial\Omega} \left(\frac{1}{3}\vec{x}\cdot\vec{n}\right) ds = \sum_{t\in T_{ri}} \boxed{\int_{\partial\Omega_t} \left(\frac{1}{3}\vec{x}\cdot\vec{n}\right) ds}$$

$$\int_{\partial\Omega_t} \frac{1}{3}(L_1\vec{x}_1 + L_2\vec{x}_2 + L_3\vec{x}_3)\cdot\vec{n} ds = \frac{1}{3}A\frac{\vec{x}_1+\vec{x}_2+\vec{x}_3}{3}\cdot\vec{n}$$

Application of Mass: Buoyancy



Force on the surface is always in normal direction

$$\vec{f} = -\rho g z \vec{n}$$

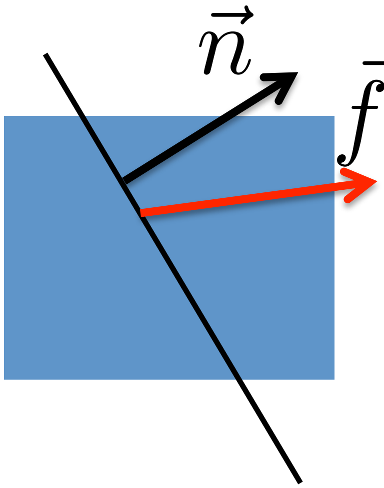
$$= \boxed{T}\vec{n}$$

Cauchy stress

$$T = -\rho g \begin{pmatrix} z & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & z \end{pmatrix}$$

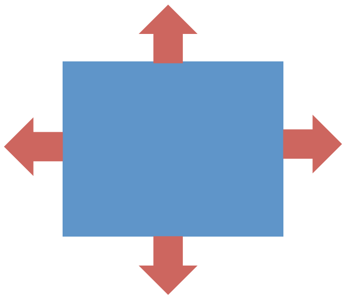
Cauchy Stress Tensor

Symmetric matrix that relate cutting plane normal $\vec{n}$ and force on the cutting plane $\vec{f}$



$$\vec{f} = \mathbf{T}\vec{n}$$

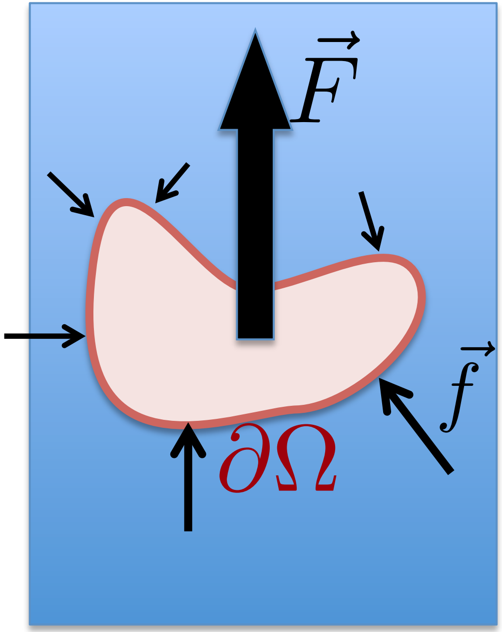
$$\mathbf{T} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \mathbf{T} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$



Application of Mass: Buoyancy

We can now proof Archimedes' principle.

$$\vec{F} = \int_{\partial\Omega} \vec{f} ds = \int_{\partial\Omega} (\mathbf{T} \cdot \vec{n}) ds = ?$$



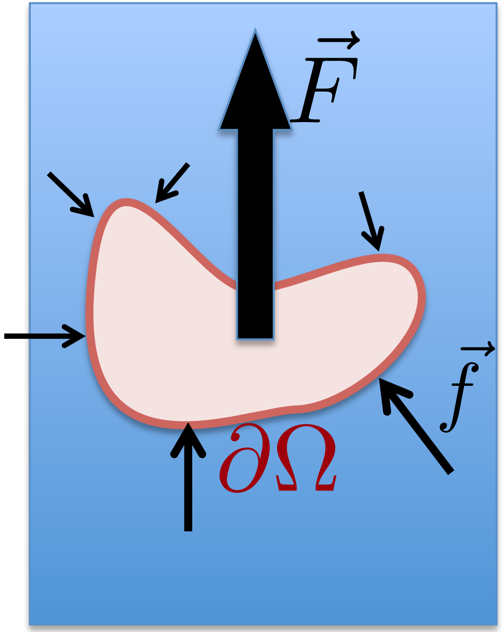
Application of Mass: Buoyancy

We can now proof Archimedes' principle.

$$\vec{F} = \int_{\partial\Omega} \vec{f} ds = \int_{\partial\Omega} (\mathbf{T} \cdot \vec{n}) ds = ?$$

The divergence theorem for tensor

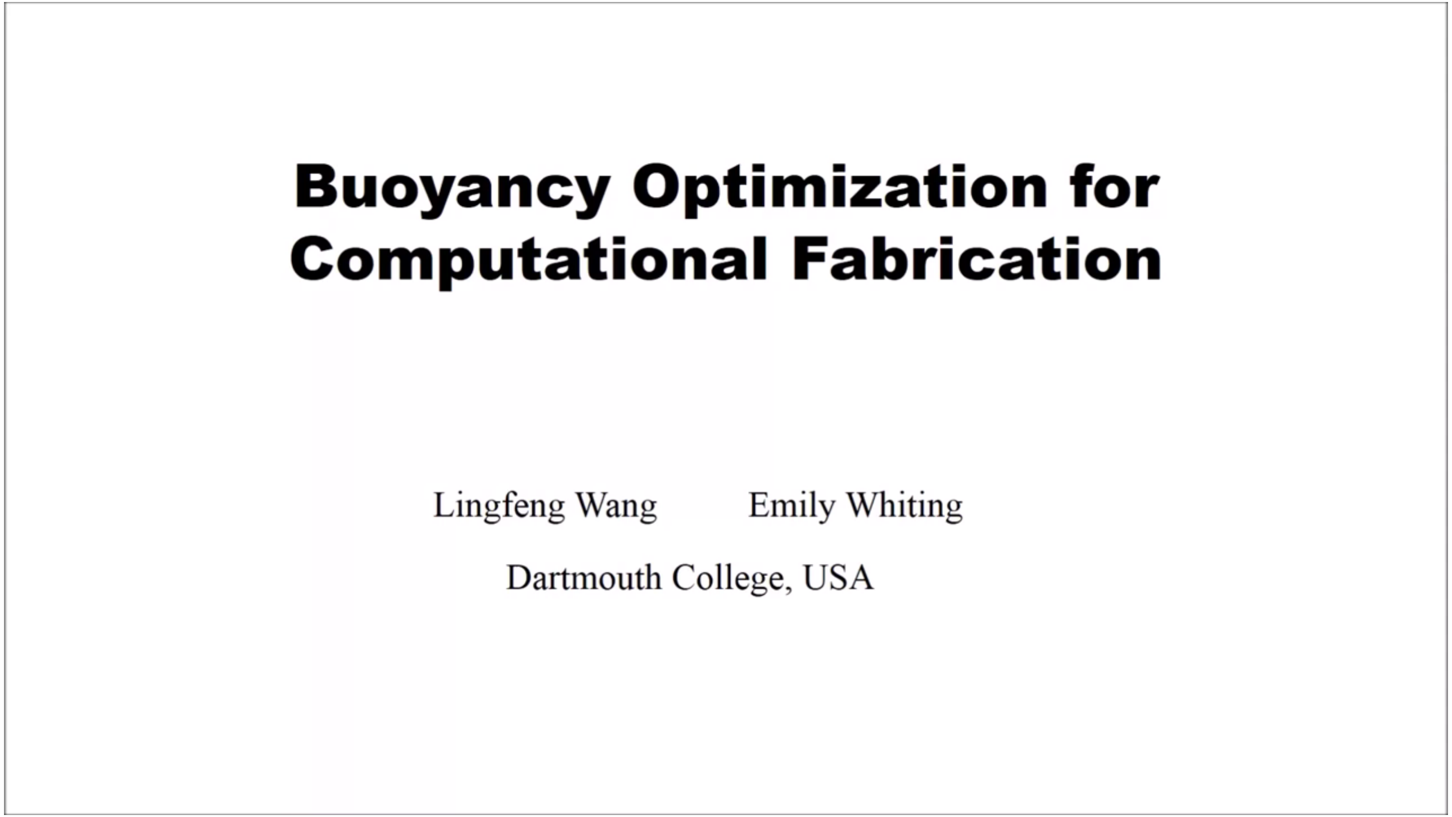
$$\int_{\Omega} (\nabla \cdot \mathbf{T}) dv = \int_{\partial\Omega} (\mathbf{T} \cdot \vec{n}) ds$$



Interesting Object: Cartesian Diver

https://www.youtube.com/watch?v=sNOXFiJ4IDU

Research in CG: Buoyancy Optimization



Buoyancy Optimization, [Wang et al. 2016] http://vcl.cs.dartmouth.edu/publications/eg-buoyancy/

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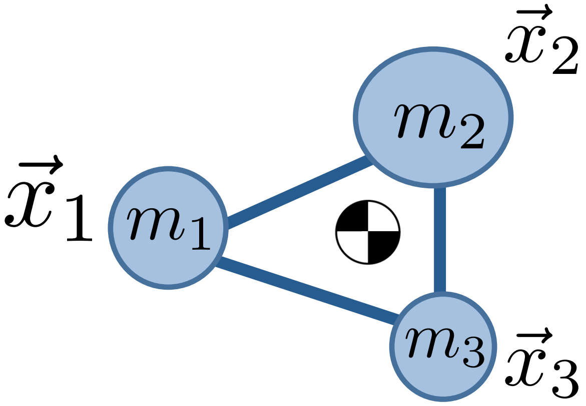
The Center of Mass

weighted average of the position

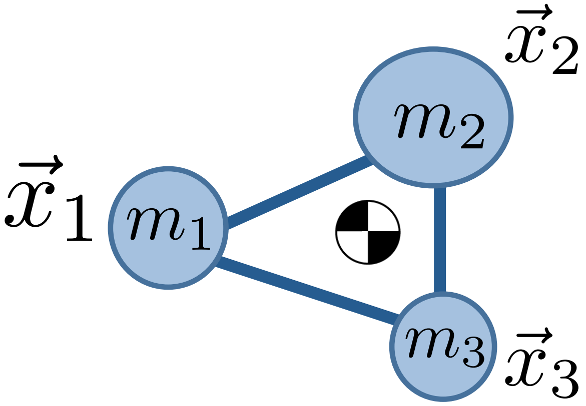
$$\vec{C} = \frac{1}{M}\sum m_i \vec{x}_i$$



The Center of Mass

weighted average of the position

$$\vec{C} = \frac{1}{M} \sum m_i \vec{x}_i$$



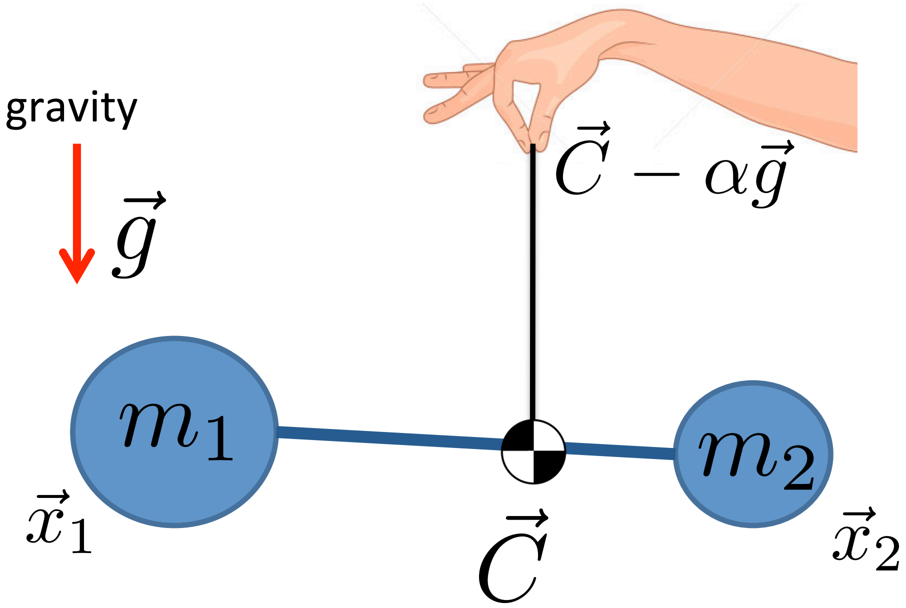
$$\vec{C} = \frac{1}{M} \int_\Omega \rho(\vec{x}) \vec{x} dv$$

Homogeneous density

$$\vec{C} = \frac{\rho}{M} \int_\Omega \vec{x} dv$$

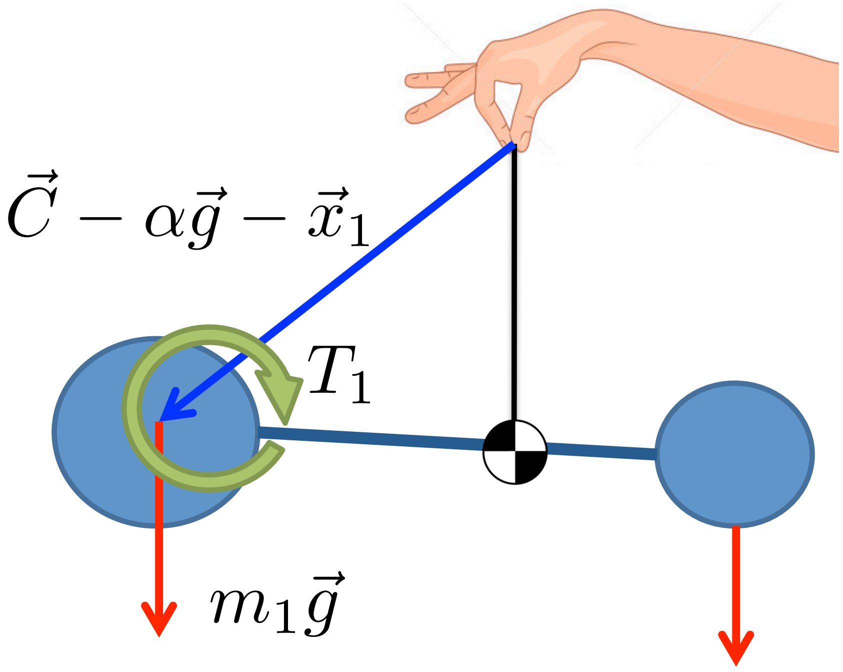
The Center of Mass and Balance

We can hang an object static by putting string above CM



The Center of Mass and Balance

We can hang an object static by putting string above CM

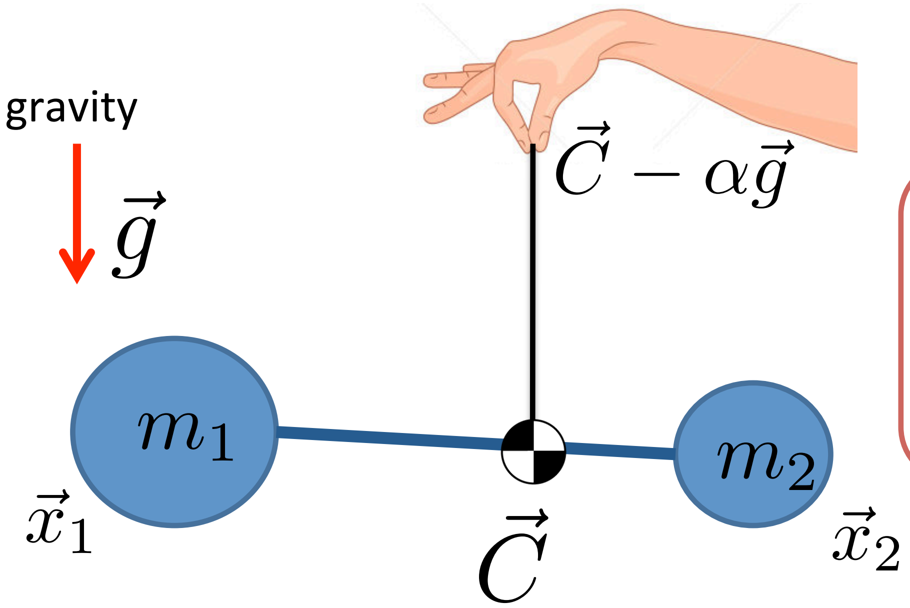


Torque from the right side of the object

$$T_1 = (m_1\vec{g}) \times (\vec{C} - \alpha\vec{g} - \vec{x}_1)$$

The Center of Mass and Balance

We can hang an object static by putting string above CM



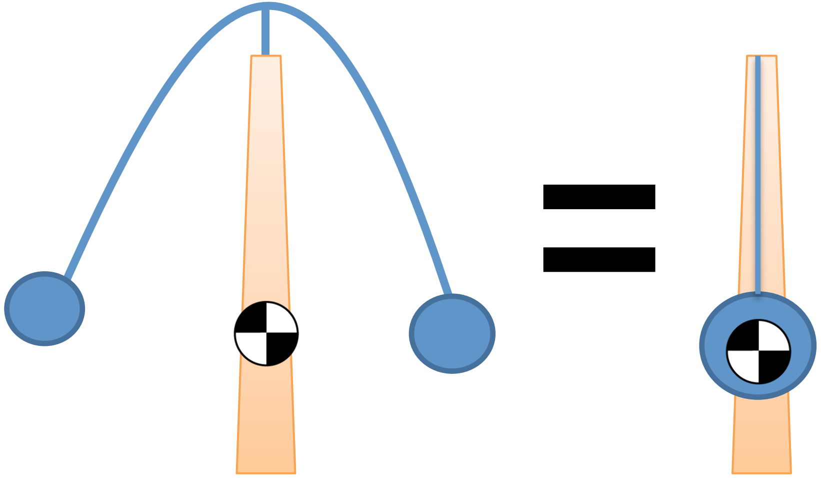
Sum of the torque is always zero

$$\sum(m_i\vec{g}) \times (\vec{C} - \alpha\vec{g} - \vec{x}_i) = 0$$

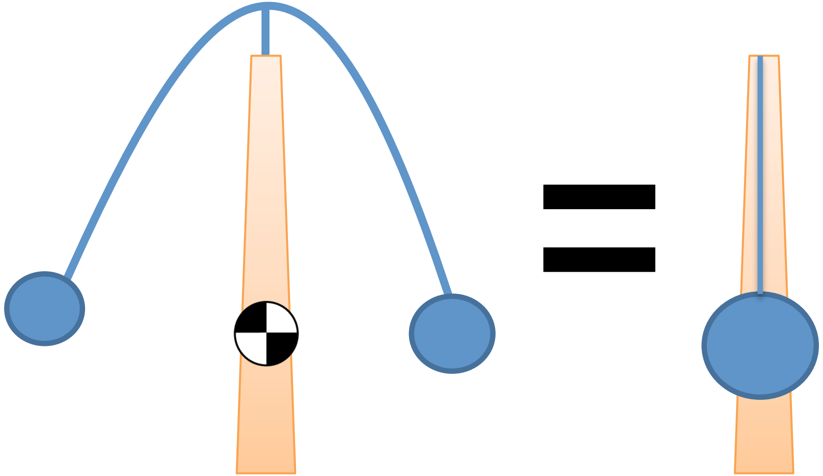
Balancing Toy

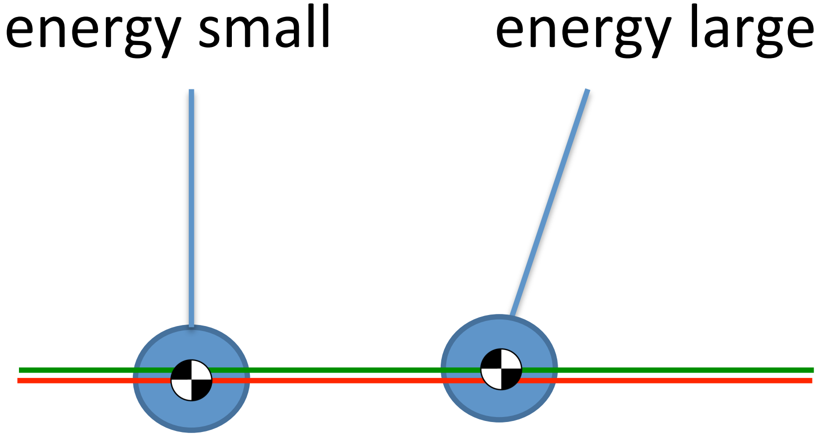
https://www.youtube.com/watch?v=wKvk-Q7U3nM

Criteria for Stability: Point Support

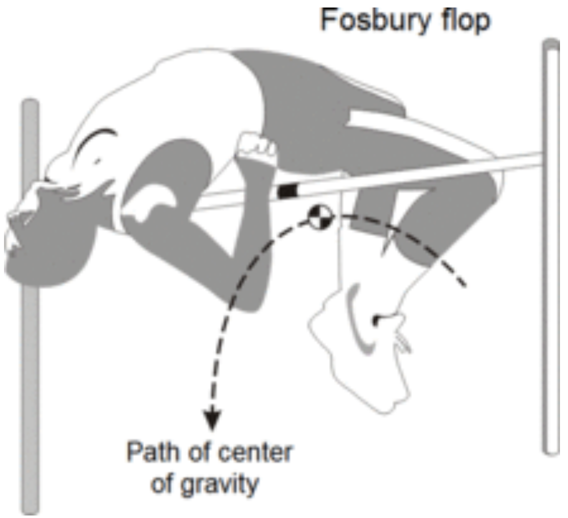


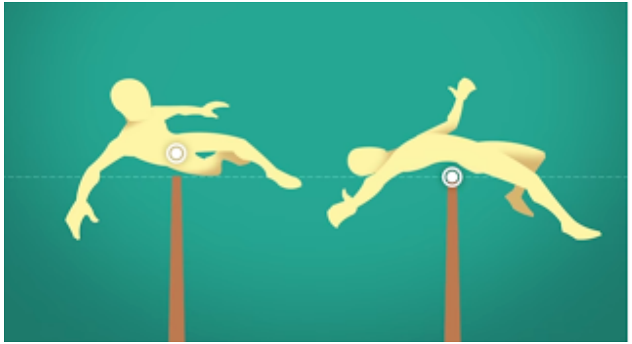
Criteria for Stability: Point Support





CM Can be Outside: The Fosbury Flop



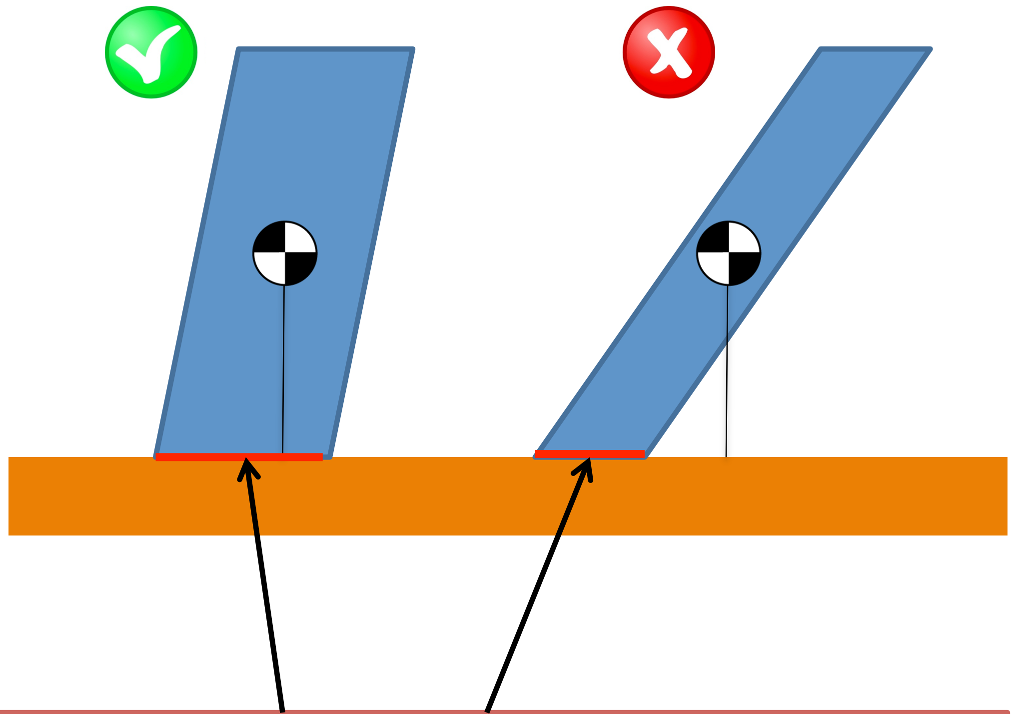


[TED Ed, https://www.youtube.com/watch?v=RaGUW1d0w8g]

1968 summer Olympic in Mexico City

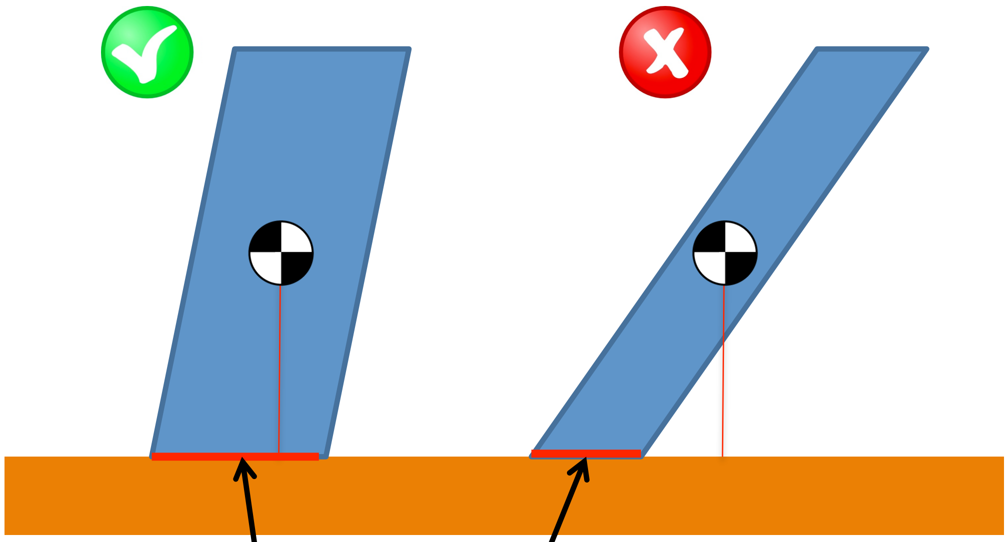
https://www.youtube.com/watch?v=Id4W6VA0uLc

Criteria for Stability: Standing



Center of mass should be above the support region

Criteria for Stability: Standing



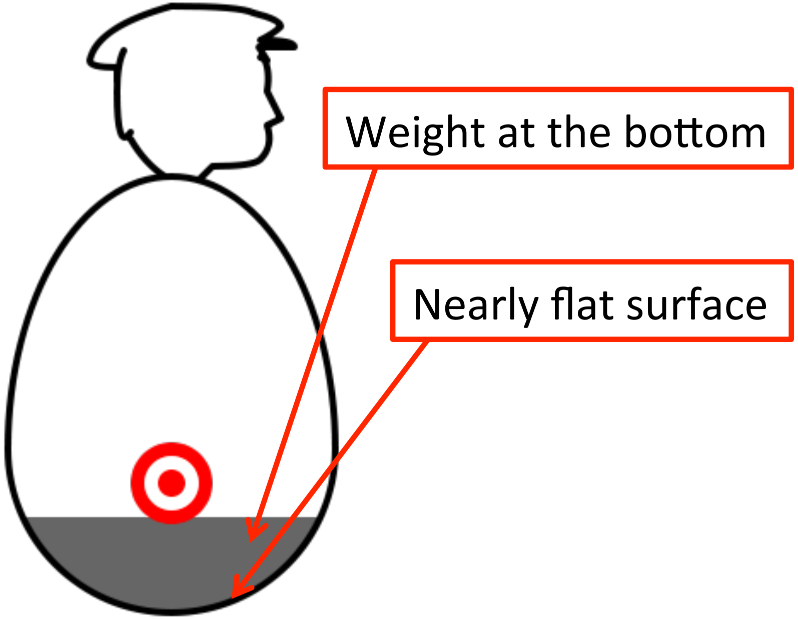
Center of mass should be above the support region

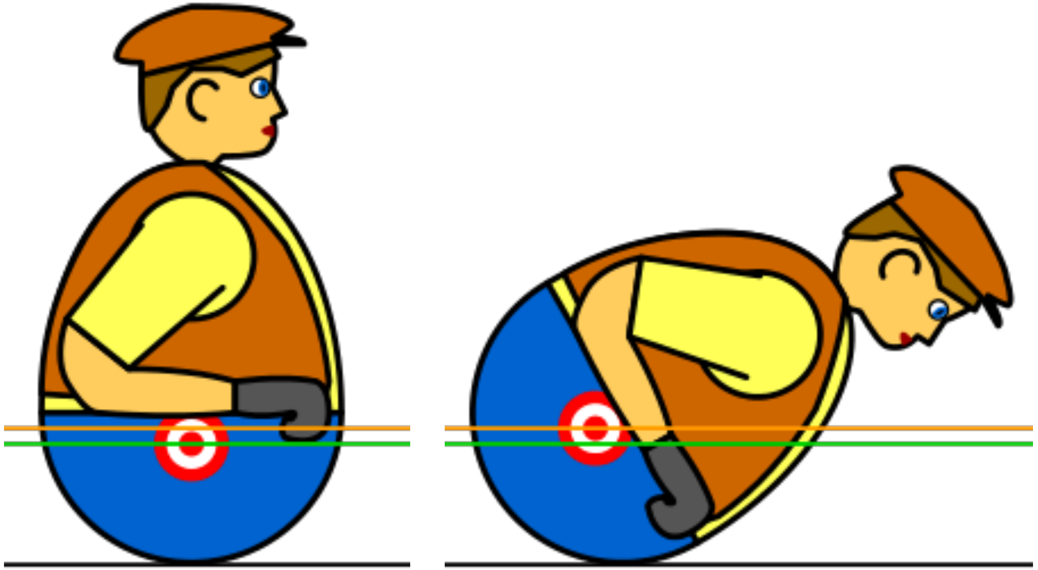
[Prevost et al. 2013]

Supporting polygon
(convex hull of touching points)

Criteria for Stability: Smooth Surface

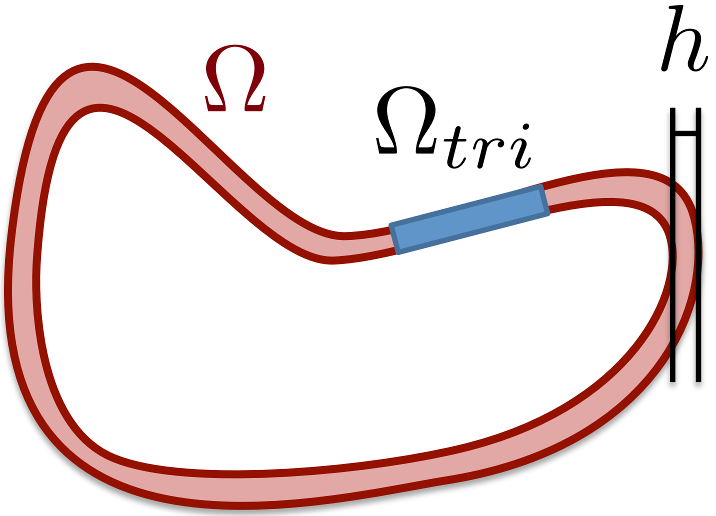
Roly-poly toy



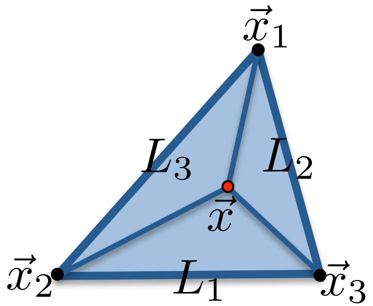


https://en.wikipedia.org/wiki/Roly-poly_toy

Center of Mass for Shell Triangle Mesh



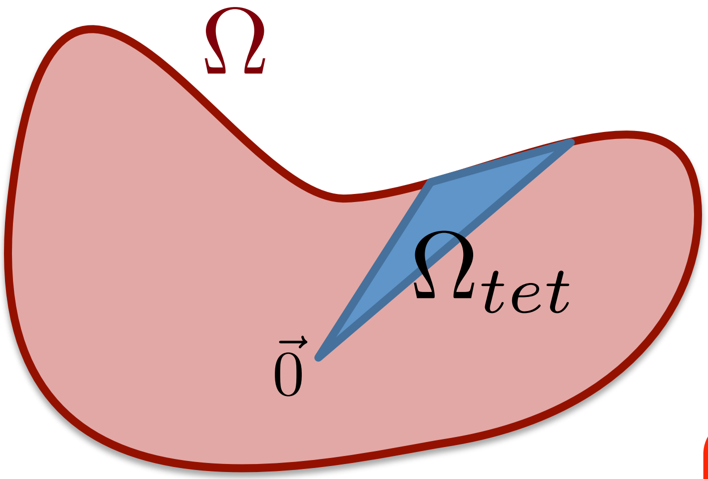
$$\int_\Omega \vec{x}\,dv = \sum_{t\in T} h \boxed{\int_{\Omega_{tri}} \vec{x}\,ds}$$



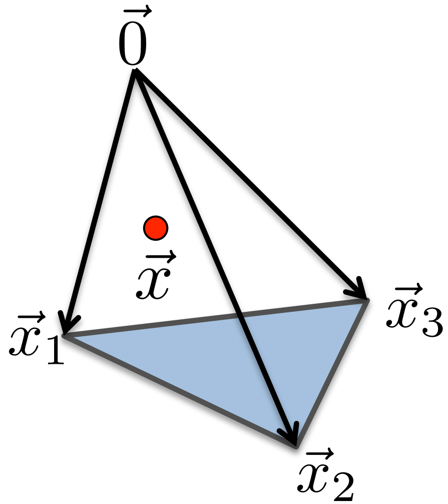
$$\boxed{\int_{\Omega_{tri}} \vec{x}\,ds} = \int_{\Omega_{tri}} (L_1\vec{x}_1 + L_2\vec{x}_2 + L_3\vec{x}_3)\,ds$$

$$= A\frac{\vec{x}_1 + \vec{x}_2 + \vec{x}_3}{3}$$

Center of Mass for **Solid** Triangle Mesh



$$\int_{\Omega} \vec{x}\,dv = \sum_{t \in T} \boxed{\int_{\Omega_{tet}} \vec{x}\,dv}$$



$$\boxed{\int_{\Omega_{tet}} \vec{x}\,dv} = \int_{\Omega_{tet}} (L_1\vec{x}_1 + L_2\vec{x}_2 + L_3\vec{x}_3 + L_4\vec{0})\,dv$$

$$= V\frac{\vec{x}_1 + \vec{x}_2 + \vec{x}_3}{4}$$

Center of Mass for Solid Triangle Mesh II

The divergence theorem for tensor

$$\int_\Omega \vec{x}\,dv = \int_\Omega \underbrace{(\nabla \cdot \mathbf{T})}_{=\vec{x}}dv = \int_{\partial\Omega} (\mathbf{T}\cdot\vec{n})ds$$

$\nabla \cdot \mathbf{T} = \vec{x}$ ➡ What's the $\mathbf{T}$ looks like?

Center of Mass for Solid Triangle Mesh Ⅱ

The divergence theorem for tensor

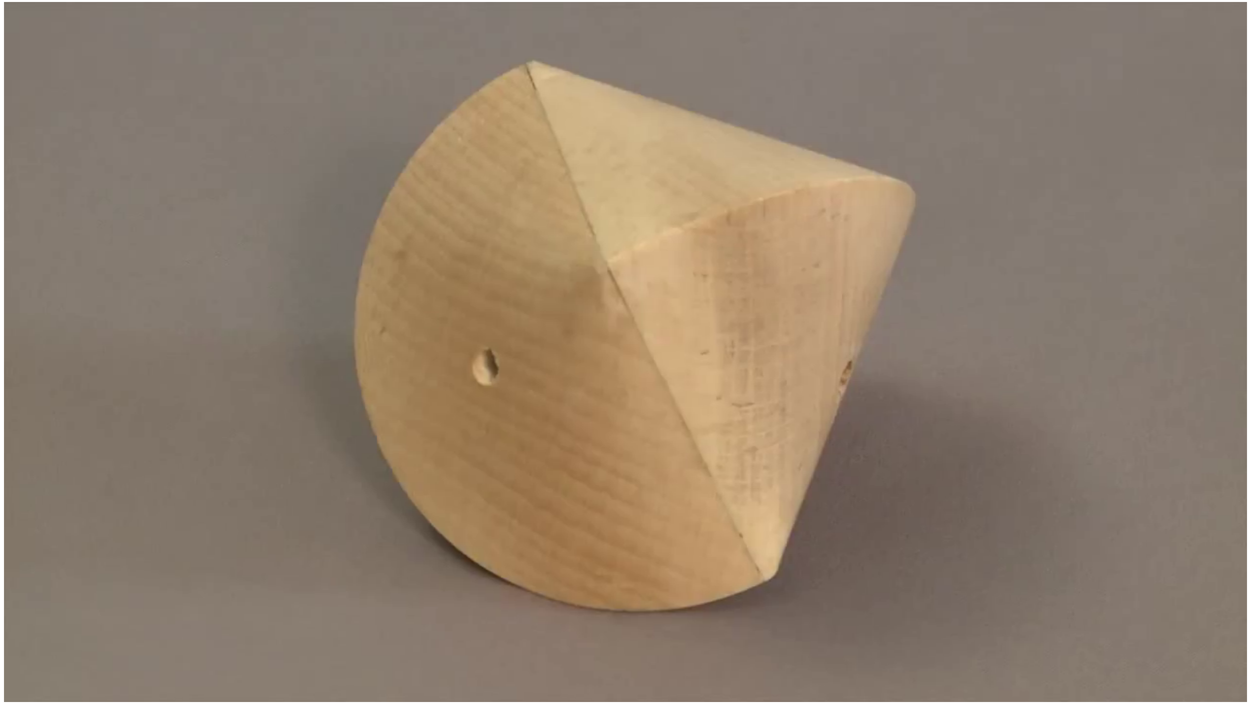
$$\int_\Omega \vec{x}dv = \int_\Omega \underbrace{(\nabla \cdot \mathbf{T})}_{=\vec{x}}dv = \int_{\partial\Omega} (\mathbf{T} \cdot \vec{n})ds$$

$$\nabla \cdot \mathbf{T} = \vec{x}$$ ➡ What's the $\mathbf{T}$ looks like?

One example:

$$\mathbf{T} = \frac{1}{2}\begin{pmatrix} x^2 & 0 & 0 \\ 0 & y^2 & 0 \\ 0 & 0 & z^2 \end{pmatrix}$$

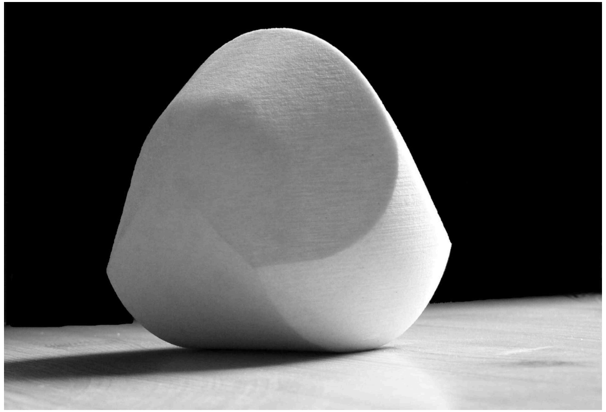
Interesting Object: Sphericon

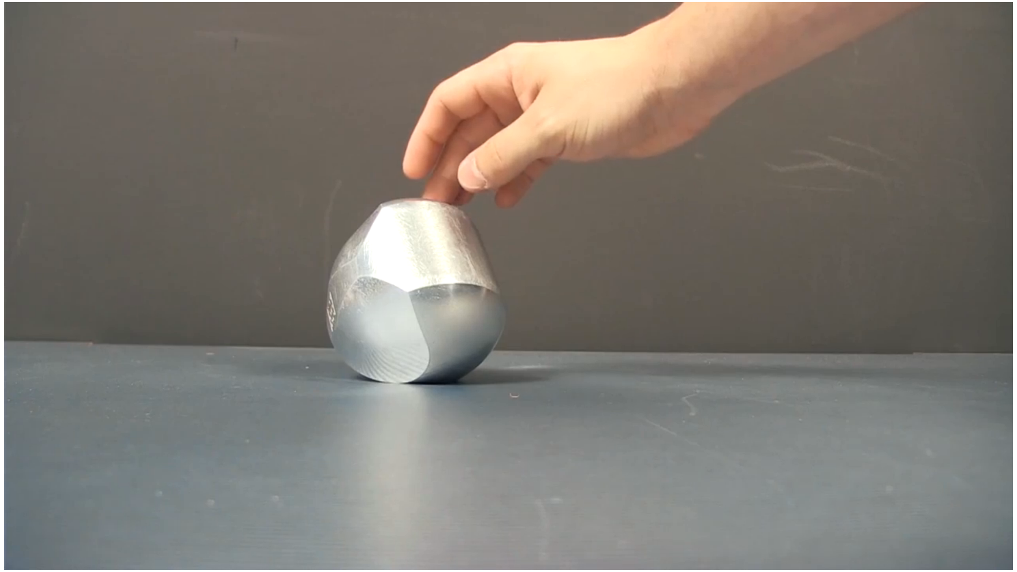


https://www.youtube.com/watch?v=L4KQmk7QEVQ

Interesting Object: Gomboc

- Convex and homogeneous object that has a single stable configuration





https://www.youtube.com/watch?v=p96YazGhlY4

Research in CG: Make it Stand

Make It Stand: Balancing Shapes for 3D Fabrication

Romain Prévost[1] Emily Whiting[1] Sylvain Lefebvre[2] Olga Sorkine-Hornung[1]
[1]ETH Zurich [2]INRIA

(contains audio)

Make It Stand, [Prevost et al. 2013], http://igl.ethz.ch/projects/make-it-stand/

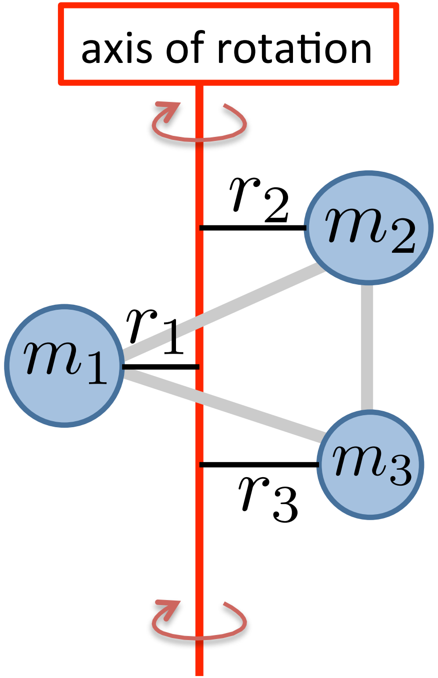
Overview

- Intro
- Integration over 3D domain
- Mass
- Center of Mass
- Moment of Inertia

The Moment of Inertia

We can define moment & energy of rotating object using MI

$$I = \sum m_i r_i^2$$



The Moment of Inertia

We can define moment & energy of rotating object using MI

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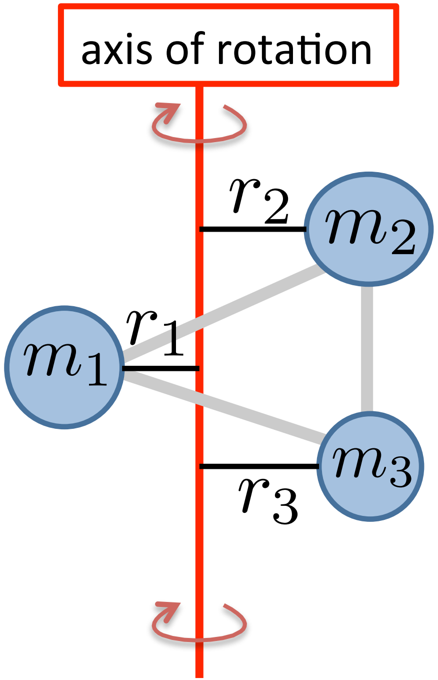
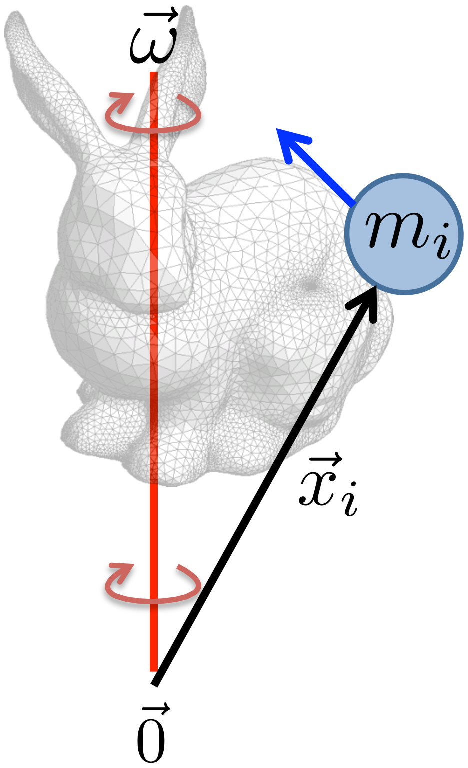


$$I = \int_\Omega \rho(\vec{x}) r(\vec{x})^2 dv \xrightarrow{\text{Homogeneous density}} I = \rho \int_\Omega r(\vec{x})^2 dv$$

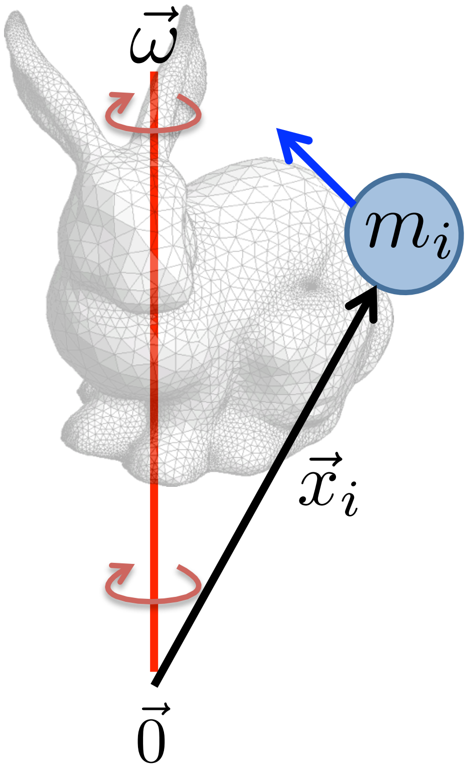
Angular Momentum and MI



angular momentum

$$\vec{L} = \sum \vec{x}_i \times (m_i \vec{v}_i)$$

Angular Momentum and MI

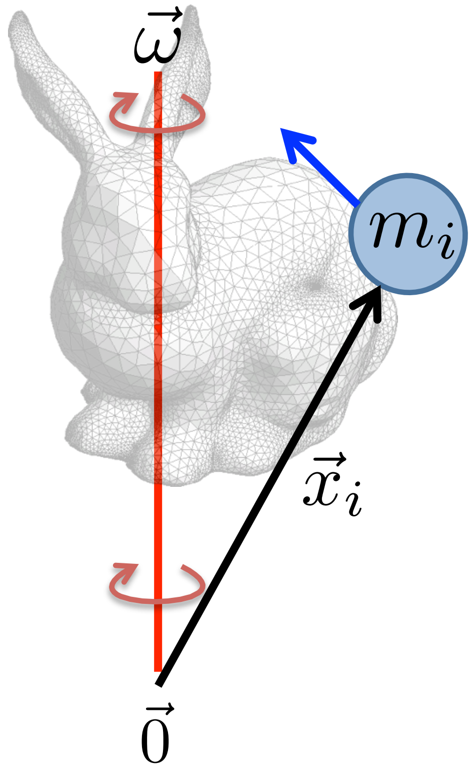


angular momentum

$$\vec{L} = \sum \vec{x}_i \times (m_i \vec{v}_i)$$

$$= \sum m_i \vec{x}_i \times (\vec{\omega} \times \vec{x}_i)$$

Angular Momentum and MI



angular momentum

$$\vec{L} = \sum \vec{x}_i \times (m_i \vec{v}_i)$$

$$= \sum m_i \vec{x}_i \times (\vec{\omega} \times \vec{x}_i)$$

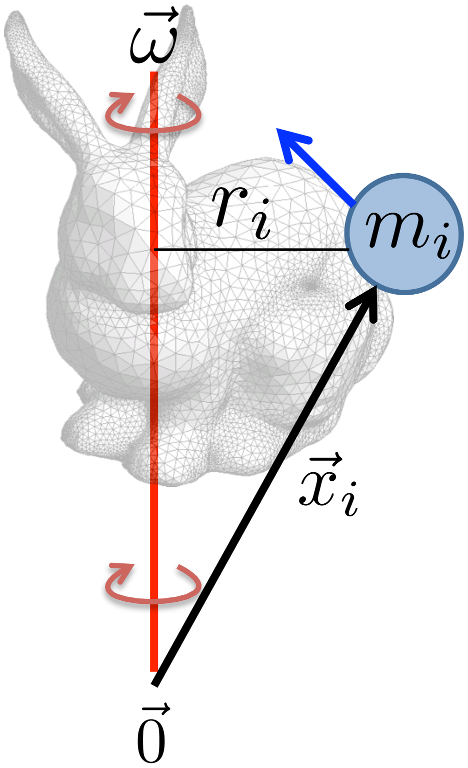
$$\boxed{\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}}$$

$$= I\vec{\omega}$$

Conservation of Angular Momentum

https://www.youtube.com/watch?v=UZlW1a63KZs

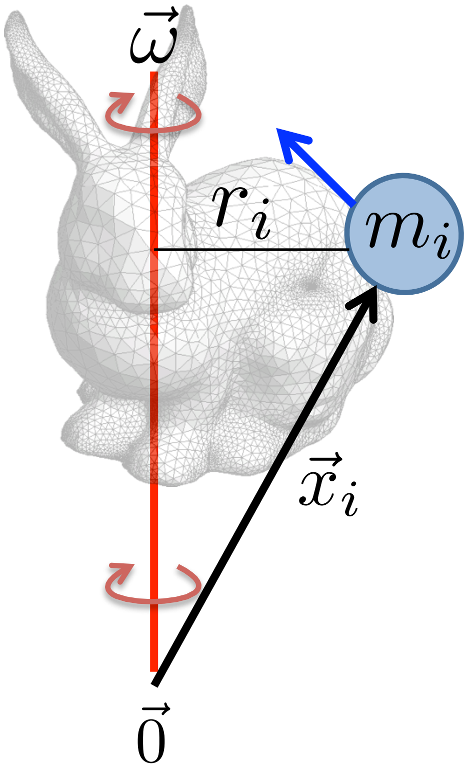
Kinetic Energy and MI



Kinetic Energy

$$E = \sum \frac{1}{2} m_i |\vec{v}_i|^2$$

Kinetic Energy and MI



Kinetic Energy

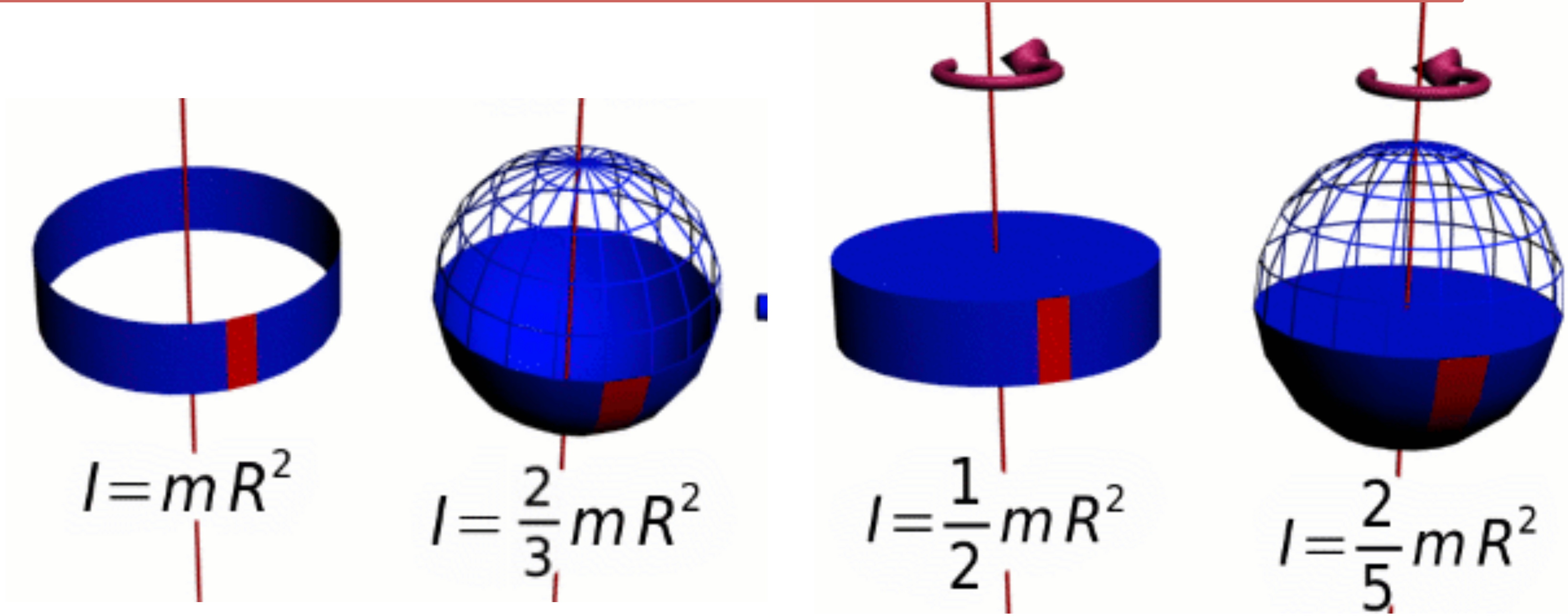
$$E = \sum \frac{1}{2} m_i |\vec{v}_i|^2$$

$$= \sum \frac{1}{2} m_i (|\vec{w}| r_i)^2$$

$$= \frac{1}{2} I |\vec{\omega}|^2$$

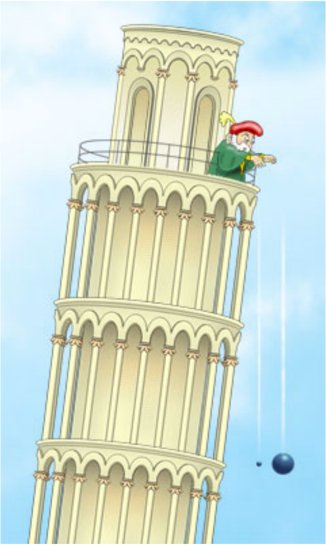
Example of Moment of Inertia

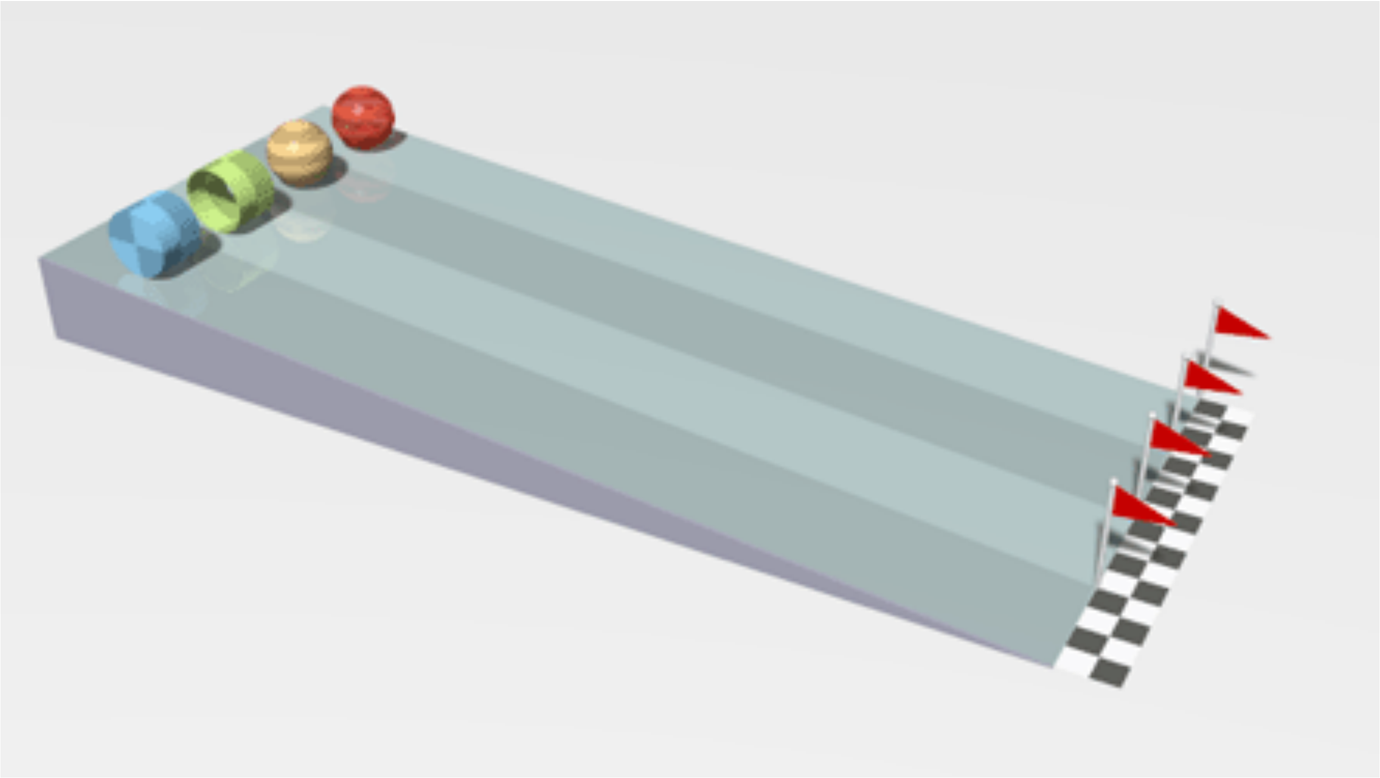
- Same weight give a different Moment of Inertia
- Putting weight outer ward give large Moment of Inertia



https://en.wikipedia.org/wiki/File:Moment_of_inertia_examples.gif

Slope Version of Galileo's Pisa Experiment

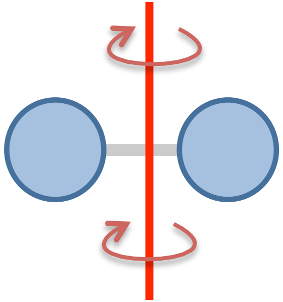




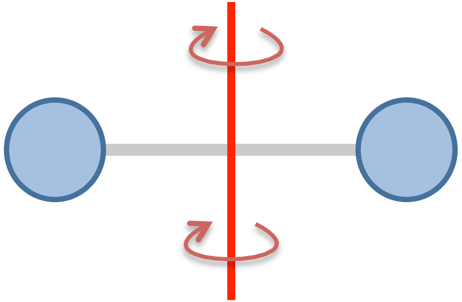
https://en.wikipedia.org/wiki/Moment_of_inertia

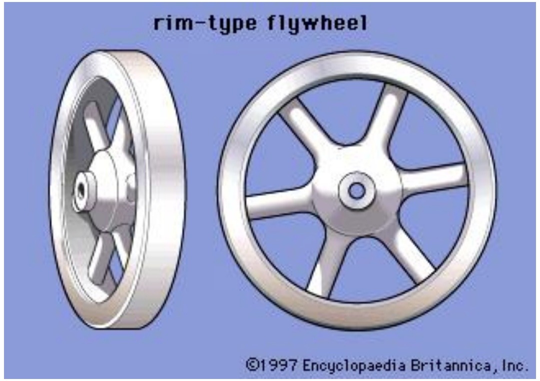
Application: Flywheel

small I



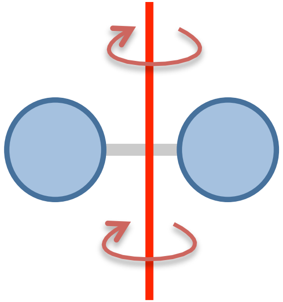
large I



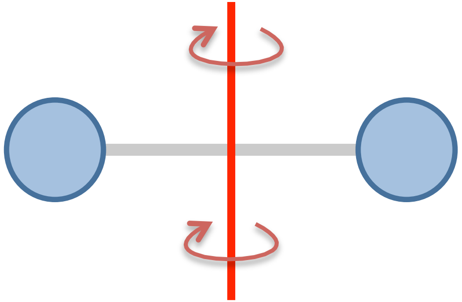


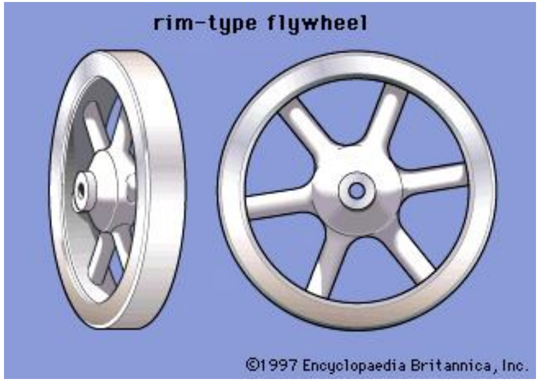
Application: Flywheel

small I

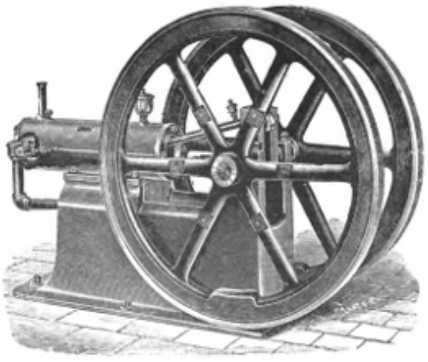


large I

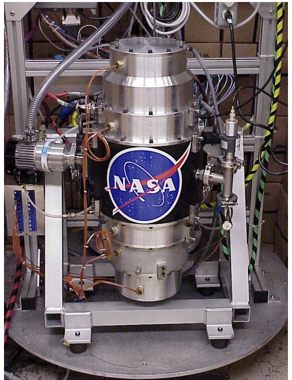




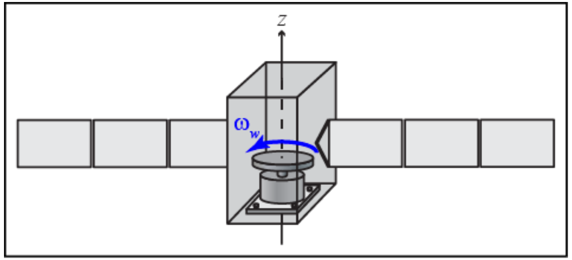
Smoothing output of engine



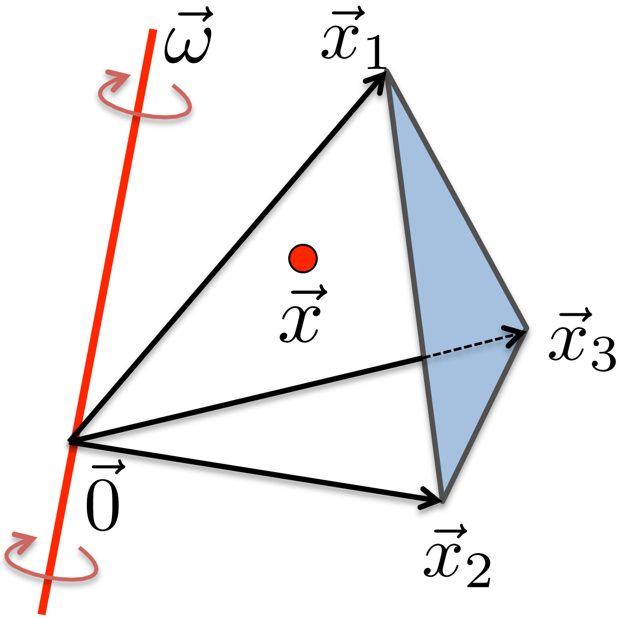
Energy storage



Reaction wheel

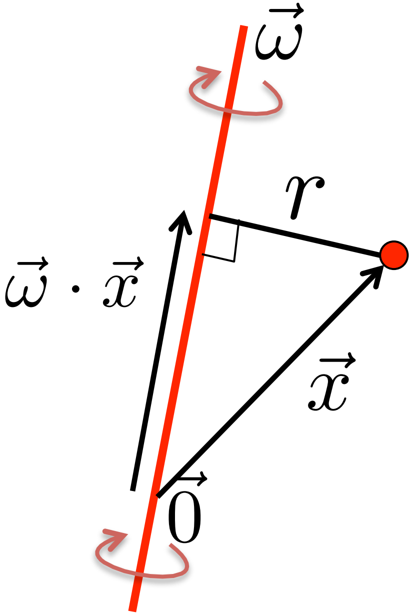


Moment of Inertia for a Tetrahedron



$$\begin{aligned}\vec{x} &= L_1\vec{x}_1 + L_2\vec{x}_2 + L_3\vec{x}_3 + L_4\vec{0} \\ &= L_i x_i\end{aligned}$$

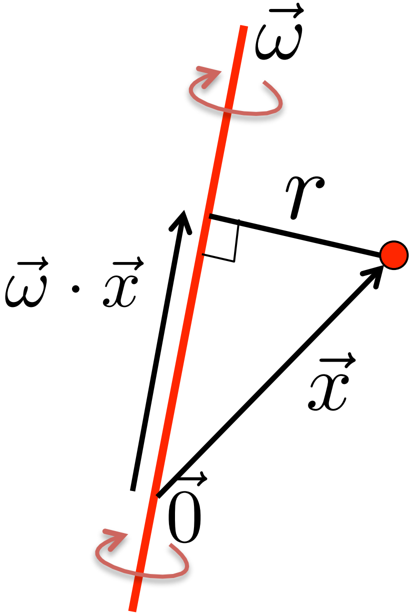
Moment of Inertia for a Tetrahedron



$$r^2 = (\vec{x} \cdot \vec{x}) - (\vec{\omega} \cdot \vec{x})^2$$

$$\begin{aligned} \vec{x} &= L_1 \vec{x}_1 + L_2 \vec{x}_2 + L_3 \vec{x}_3 + L_4 \vec{0} \\ &= L_i \vec{x}_i \end{aligned}$$

Moment of Inertia for a Tetrahedron



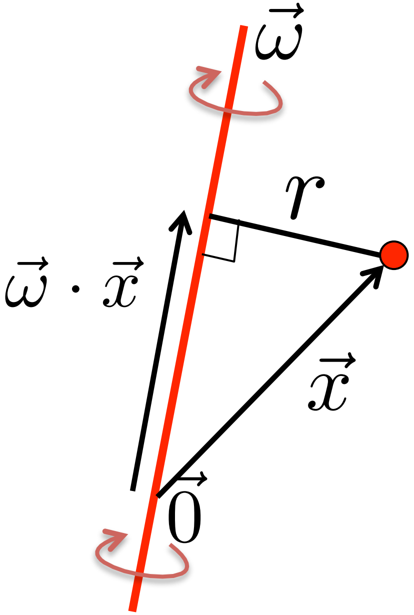
$$r^2 = (\vec{x} \cdot \vec{x}) - (\vec{\omega} \cdot \vec{x})^2$$

$$= (L_i \vec{x}_i) \cdot (L_j \vec{x}_j) - (L_i \vec{x}_i \cdot \vec{\omega}) \cdot (L_j \vec{x}_j \cdot \vec{\omega})$$

$$\vec{x} = L_1 \vec{x}_1 + L_2 \vec{x}_2 + L_3 \vec{x}_3 + L_4 \vec{0}$$

$$= L_i \vec{x}_i$$

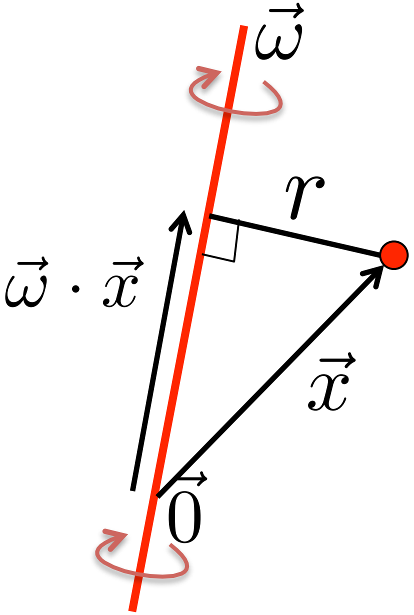
Moment of Inertia for a Tetrahedron



$$
\begin{aligned}
r^2 &= (\vec{x} \cdot \vec{x}) - (\vec{\omega} \cdot \vec{x})^2 \\
&= (L_i \vec{x}_i) \cdot (L_j \vec{x}_j) - (L_i \vec{x}_i \cdot \vec{\omega}) \cdot (L_j \vec{x}_j \cdot \vec{\omega}) \\
&= L_i L_j (\vec{x}_i \cdot \vec{x}_j) - L_i L_j (\vec{x}_i \cdot \vec{\omega})(\vec{x}_j \cdot \vec{\omega})
\end{aligned}
$$

$$
\begin{aligned}
\vec{x} &= L_1 \vec{x}_1 + L_2 \vec{x}_2 + L_3 \vec{x}_3 + L_4 \vec{0} \\
&= L_i \vec{x}_i
\end{aligned}
$$

Moment of Inertia for a Tetrahedron



$$\vec{x} = L_1 \vec{x}_1 + L_2 \vec{x}_2 + L_3 \vec{x}_3 + L_4 \vec{0}$$
$$= L_i \vec{x}_i$$

$$r^2 = (\vec{x} \cdot \vec{x}) - (\vec{\omega} \cdot \vec{x})^2$$
$$= (L_i \vec{x}_i) \cdot (L_j \vec{x}_j) - (L_i \vec{x}_i \cdot \vec{\omega}) \cdot (L_j \vec{x}_j \cdot \vec{\omega}$$
$$= \boxed{L_i L_j} (\vec{x}_i \cdot \vec{x}_j) - L_i L_j (\vec{x}_i \cdot \vec{\omega})(\vec{x}_j \cdot \vec{\omega})$$

integration

$$\boxed{\int_V L_1^a L_2^b L_3^c L_4^d dv = \frac{a!b!c!d!3!}{(a+b+c+d+3)!} V}$$

$$\frac{V}{20} (i \neq j), \quad \frac{V}{10} (i = j)$$

Research in CG: Spin-It

Spin-It: Optimizing Moment of Inertia for Spinnable Objects

Moritz Bächer, Disney Research Zurich
Emily Whiting, ETH Zurich
Bernd Bickel, Disney Research Zurich
Olga Sorkine-Hornung, ETH Zurich

Disney Research ETH zürich igl

Spin-It, [Baecher et al. 2014], https://www.disneyresearch.com/project/spin-it/

Take aways

- Integration formula: divide region into simplexes
- Switch a volume/surface integration using the divergence theorem
- Three moments of shapes
  - mass, center of mass, moment of inertia
- Many physics are given by the moments
  - floating, standing, rotation