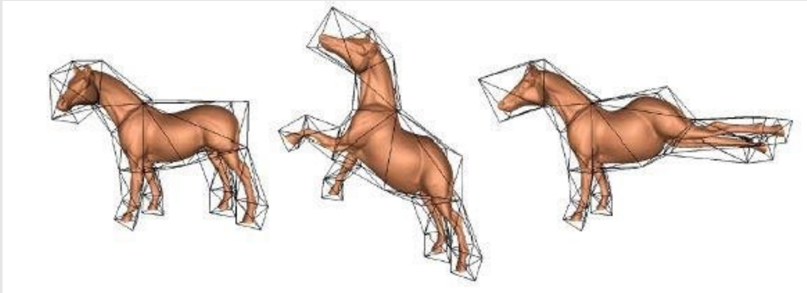
# Advanced Interpolation

# Application of Interpolation

***Cage-based deformation***

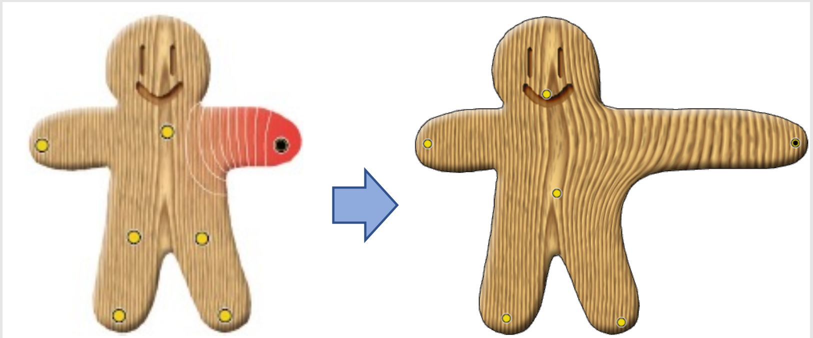
Interpolating positions in a polygon



[Ju et al. 2005]

***Rigged-based deformation***

Interpolating rigging weight inside mesh

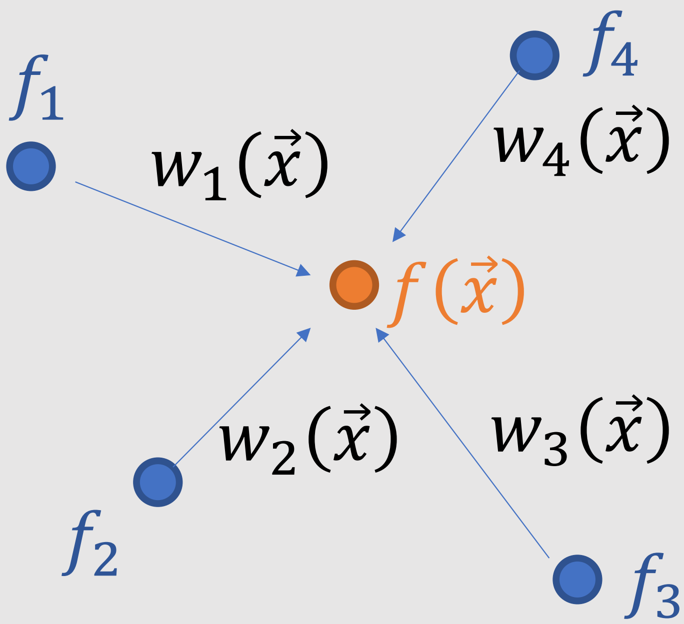


[Jacobson et al. 2005]

T. Ju, S. Schaefer, and J. Warren. ***Mean value coordinates for closed triangular meshes***. SIGGRAPH 2005

A. Jacobson, I. Baran, J. Popović, and O. Sorkine. ***Bounded biharmonic weights for real-time deformation***. SIGGRAPH 2011

# What is the Requirements for the Weights?



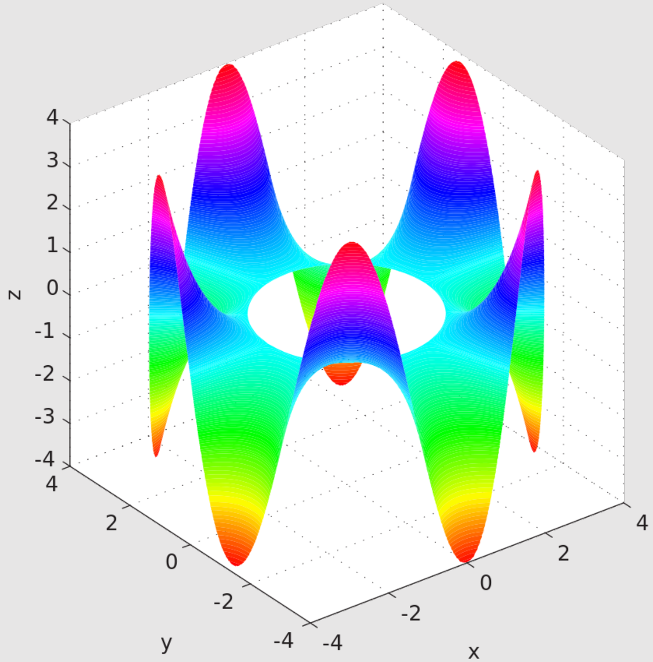

$$f(\vec{x}) = \sum_i w_i(\vec{x}) f_i$$

- Weight $w_i(\vec{x})$ must change smoothly

→ $w_i(\vec{x})$ is a harmonic function (e.g., solution of Laplace equation)

# The Property of Laplace Equation

By definition Laplace equation has no peak ($\nabla^2 \phi = 0$)



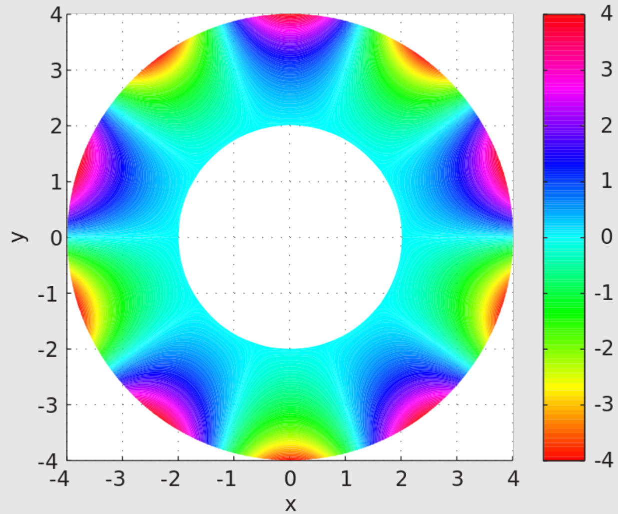
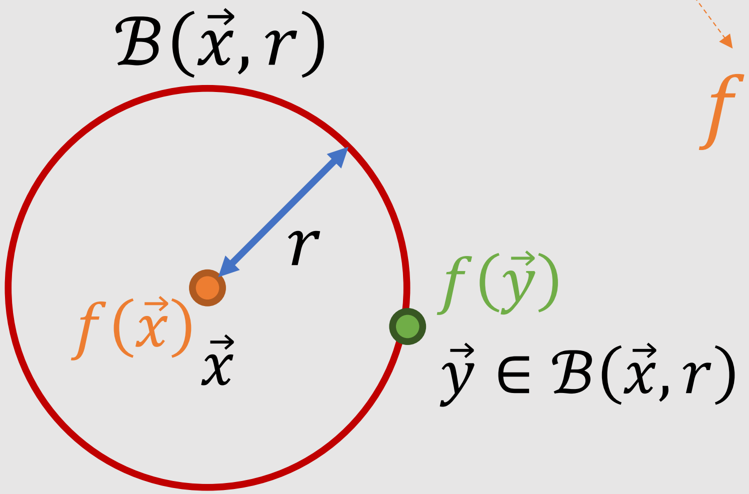




Image Credit: Fourthirtytwo @ Wikipedia

# Laplace Equation: Mean Value Property

The solution of the Laplace equation at $\vec{x}$ is average of the solution on the circle around $\vec{x}$




The sum of the solution on $\mathcal{B}(\vec{x}, r)$

Solution at $\vec{x}$

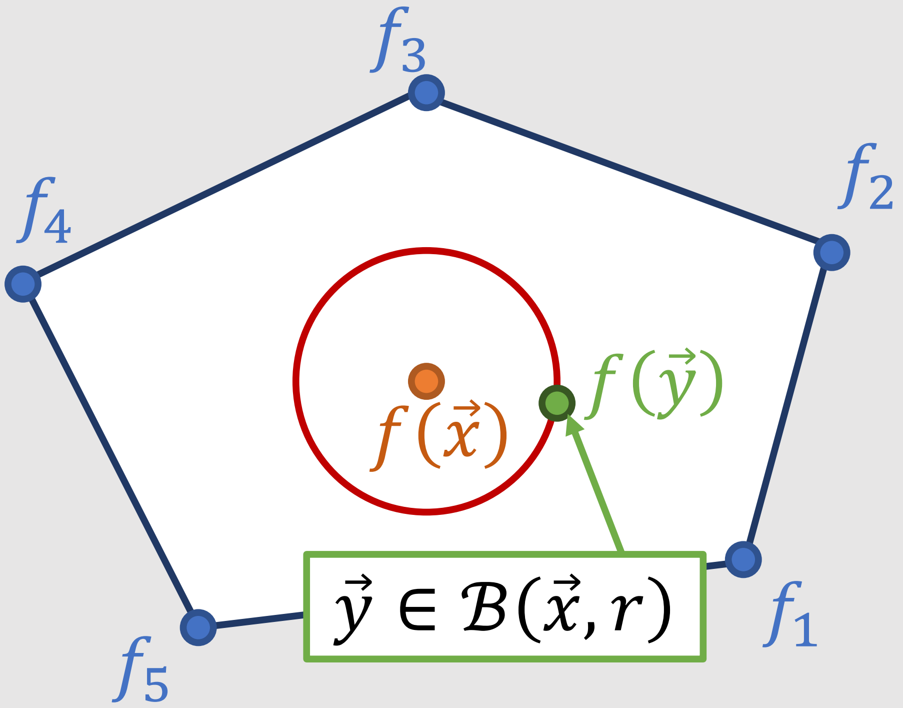
$$f(\vec{x}) = \frac{\int_{\vec{y} \in \mathcal{B}(\vec{x},r)} f(\vec{y}) \mathrm{d}\vec{y}}{\int_{\vec{y} \in \mathcal{B}(\vec{x},r)} \mathrm{d}\vec{y}} \quad \text{for arbitrary } r > 0$$

circumference of $\mathcal{B}(\vec{x}, r)$

Wikipedia: https://en.wikipedia.org/wiki/Harmonic_function#The_mean_value_property

# Mean Value Coordinate [Floater 2003]

- Interpolation inside polygon using the mean value property

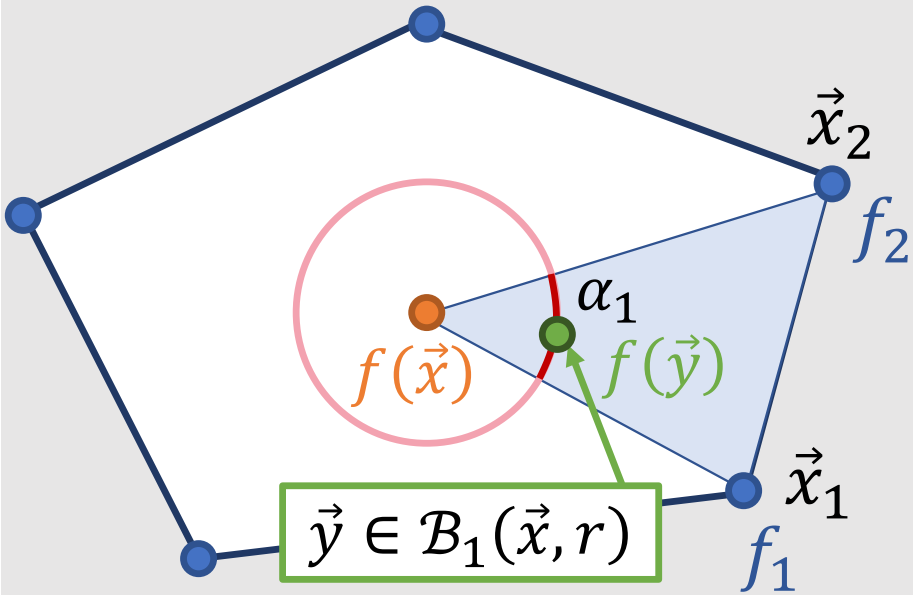



$$f(\vec{x}) = \frac{\int_{\vec{y} \in \mathcal{B}(\vec{x},r)} f(\vec{y}) \mathrm{d}\vec{y}}{\underbrace{\int_{\vec{y} \in \mathcal{B}(\vec{x},r)} \mathrm{d}\vec{y}}_{2\pi r}}$$

Michael S. Floater. 2003. Mean value coordinates. Comput. Aided Geom. Des. 20, 1 (March 2003), 19–27.

# Mean Value Coordinate [Floater 2003]

- Assume $f(\vec{y})$ is a linear interpolation inside the triangle $(\vec{x}_1, \vec{x}_2, \vec{x}\ )$




$$f(\vec{y}) = L_1 f_1 + L_2 f_2 + (1 - L_1 - L_2) f(\vec{x})$$

Integration on the arc $\mathcal{B}_1(\vec{x}, r)$

$$\int_{\vec{y} \in \mathcal{B}_1(\vec{x}, r)} f(\vec{y}) \mathrm{d}\vec{y}$$

$$= r\alpha_1 f(\vec{x}) + r^2 \tan\left(\frac{\alpha_1}{2}\right)\left[\frac{f(\vec{x}) - f_1}{|\vec{x} - \vec{x}_1|} - \frac{f(\vec{x}) - f_2}{|\vec{x} - \vec{x}_2|}\right]$$


Michael S. Floater. 2003. Mean value coordinates. Comput. Aided Geom. Des. 20, 1 (March 2003), 19–27.

# Mean Value Coordinate [Floater 2003]

- Closed-form approximate solution in a convex polygon

$$f(\vec{x}) = \frac{\int_{\vec{y}\in\mathcal{B}(\vec{x},r)} f(\vec{y})\mathrm{d}\vec{y}}{\int_{\vec{y}\in\mathcal{B}(\vec{x},r)} \mathrm{d}\vec{y}} = \frac{1}{2\pi r}\sum_{e\in Edges}\int_{\vec{y}\in\mathcal{B}_e(\vec{x},r)} f(\vec{y})\mathrm{d}\vec{y}$$

$$\int_{\vec{y}\in\mathcal{B}_e(\vec{x},r)} f(\vec{y})\mathrm{d}\vec{y} = r\alpha_e f(\vec{x}) + r^2\tan\left(\frac{\alpha_e}{2}\right)\left[\frac{f(\vec{x}) - f_e}{|\vec{x}-\vec{x}_e|} - \frac{f(\vec{x}) - f_{e+1}}{|\vec{x}-\vec{x}_{e+1}|}\right]$$

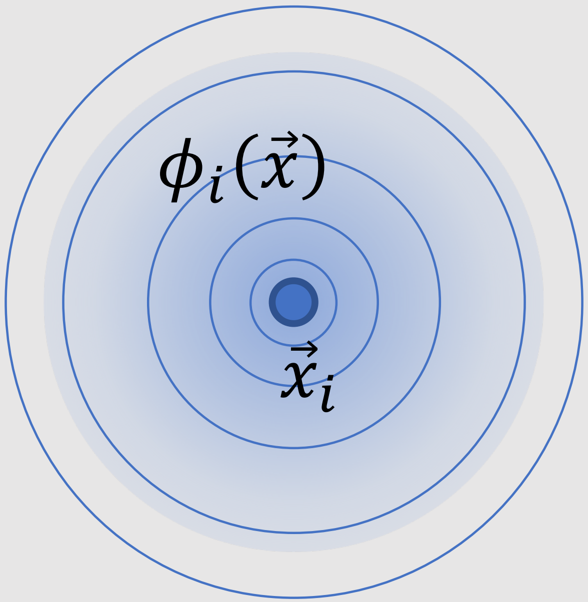
$$f(\vec{x}) = \frac{\sum_{p\in Points} w_p f_p}{\sum_{p\in Points} w_p}, \qquad where\ w_p = \frac{\tan(\alpha_p/2) + \tan(\alpha_{p+1}/2)}{|\vec{x}-\vec{x}_p|}$$

# Radial Based Function Interpolation

# Radial Based Function （放射基底関数）

- The function depends only on radius (rotational symmetry)

$$\phi_i(\vec{x}) = \phi(|\vec{x} - \vec{x}_i|)$$




The contours line of $\phi_i(\vec{x})$
are circles centers at $\vec{x}_i$

# Radial Based Function（放射基底関数）

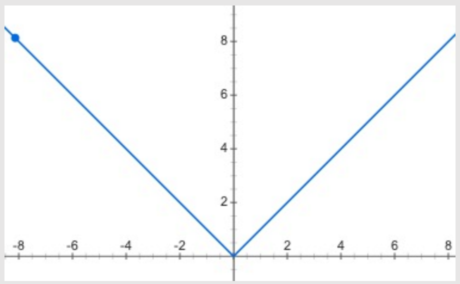
- The function depends only on radius (rotational symmetry)

$$\phi_i(\vec{x}) = \phi(|\vec{x} - \vec{x}_i|)$$

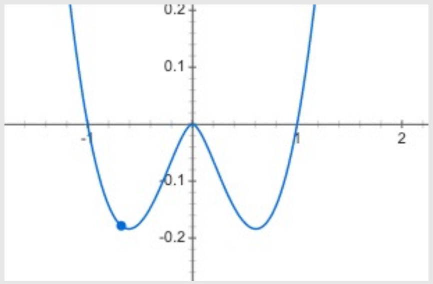
Typical choice of $\phi(r)$

-> Fundamental solution of Biharmonic equation $\nabla^4 \phi = 0$

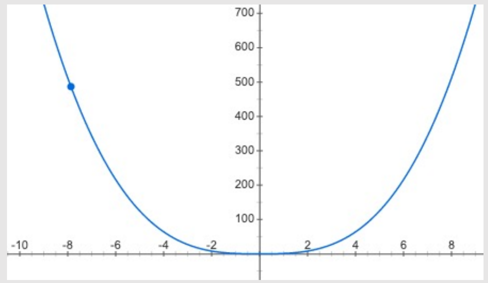
1D: $\phi(r) = r$




2D: $\phi(r) = r^2 \ln(r)$




3D: $\phi(r) = r^3$

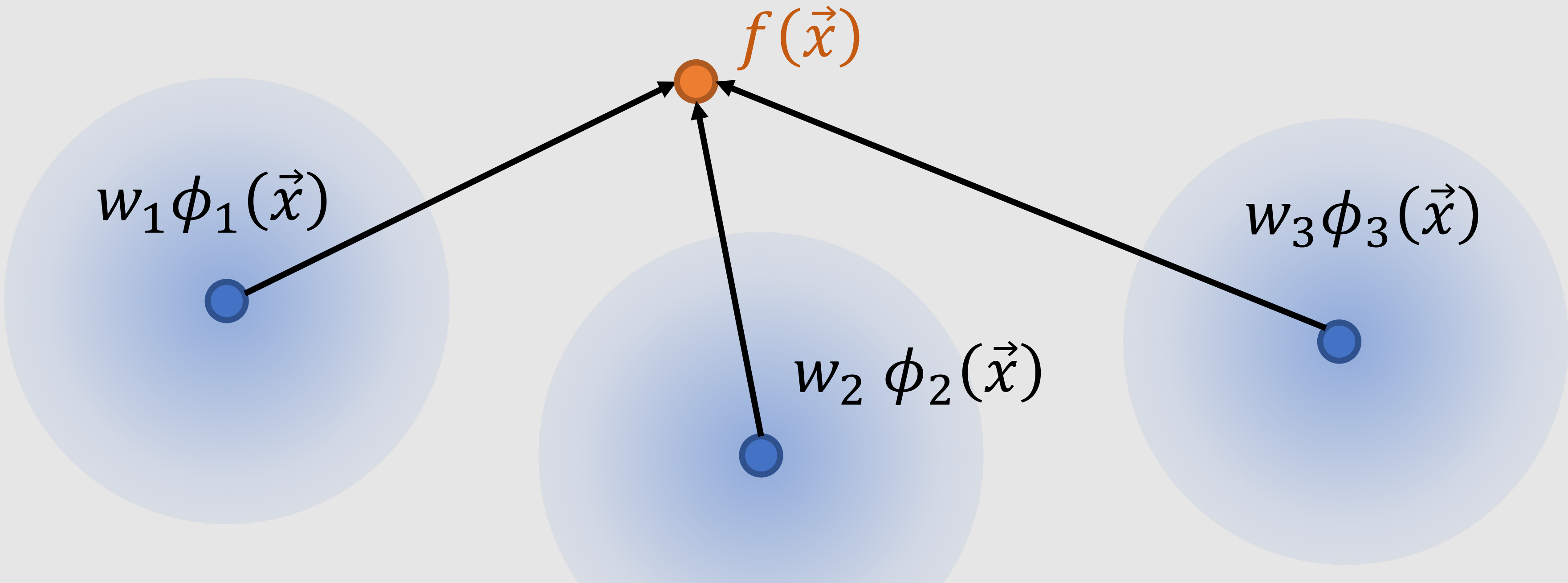


# Radial Based Function Interpolation

- The interpolation is weighted sum of the RBFs

$$f(\vec{x}) = \sum_{i \in N} w_i \phi(|\vec{x} - \vec{x}_i|)$$

How we can compute $w_i$ ?



# Radial Based Function Interpolation

- The interpolation is weighted sum of the RBFs

$$f(\vec{x}) = \sum_{i \in N} w_i \phi(|\vec{x} - \vec{x}_i|)$$

How we can compute $w_i$ ?

- Weighs are computed to satisfy constraints $f(\vec{x}_i) = f_i$

$$\underbrace{\begin{bmatrix} \phi(|\vec{x}_1 - \vec{x}_1|) & \phi(|\vec{x}_1 - \vec{x}_2|) & \cdots & \phi(|\vec{x}_1 - \vec{x}_N|) \\ \phi(|\vec{x}_2 - \vec{x}_1|) & \phi(|\vec{x}_2 - \vec{x}_2|) & \cdots & \phi(|\vec{x}_2 - \vec{x}_N|) \\ \vdots & \vdots & \ddots & \vdots \\ \phi(|\vec{x}_N - \vec{x}_1|) & \phi(|\vec{x}_N - \vec{x}_2|) & \cdots & \phi(|\vec{x}_N - \vec{x}_N|) \end{bmatrix}}_{K} \underbrace{\begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{pmatrix}}_{\vec{w} = K^{-1}\vec{f}} = \underbrace{\begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_N \end{pmatrix}}_{\vec{f}}$$

# Reference

- **Scattered Data Interpolation for Computer Graphics,** Ken Anjyo, J.P. Lewis, Frédéric Pighin, *Proceeding SIGGRAPH '14 ACM SIGGRAPH 2014 Courses Article No. 27*

https://olm.co.jp/rd/research_event/scattered-data-interpolation-for-computer-graphics/?lang=en