Boundary Element Method
Helmhotlz Equatoin

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1 Boundary Formulation of Acoustics

To make the paper self-contained, we briefly explain the boundary formulation of the Helmholtz equation. We refer readers to the book [?] for more details of the BEM implementation. The Helmholtz equation (1) has a kernel

$$G(\mathbf{x},\mathbf{y}) = \frac{\exp(+ikr)}{4\pi r}, \quad \text{where } r = ||\mathbf{x}-\mathbf{y}||, \tag{1}$$

which is the fundamental solution to the Dirac delta function $\delta(\mathbf{x}-\mathbf{y})$. Using this kernel function, the second Stoke's theorem leads to the equation which the sound pressure on the surface $p(\mathbf{x})$ needs to satisfy

$$\frac{\Omega(\mathbf{x})}{4\pi} p(\mathbf{x}) + \int_S \frac{\partial G(\mathbf{x},\mathbf{y})}{\partial \mathbf{n}(\mathbf{y})} p(\mathbf{y}) ds(\mathbf{y}) = G(\mathbf{x},\mathbf{x}_{src}), \quad \mathbf{x} \in S, \tag{2}$$

where the $\partial G(\mathbf{x},\mathbf{y})/\partial \mathbf{n}(\mathbf{y})$ derivative the kernel with respect to change of $\mathbf{y} \in S$ in the normal direction of the surface is

$$\frac{\partial G(\mathbf{x},\mathbf{y})}{\partial \mathbf{n}(\mathbf{y})} = \frac{\exp(+ikr)}{4\pi r^2} (1 - ikr) \frac{(\mathbf{x}-\mathbf{y}) \cdot \mathbf{n}}{||\mathbf{x}-\mathbf{y}||}. \tag{3}$$

The $\Omega(\mathbf{x})$ is a solid angle which takes 2π on a smooth surface, and is computed for triangle mesh using a formula presented in [?]. In our implementation, the sound pressure is stored at the vertices of a triangle mesh and linearly interpolated over the triangle faces. We discretize equation (2) using a typical collocation method, which formulates a linear system (3) by satisfying the equation at every vertex. We use a fifth-order Gaussian quadrature to compute this surface integration.

Once the reflection pressure at the vertices $\mathbf{p}$ in (3) is solved, the pressure value at the observation point $\mathbf{x}_{obs}$ inside medium Ω is computed with the surface integration

$$p(\mathbf{x}_{obs}) = -\int_S \frac{\partial G(\mathbf{x}_{obs},\mathbf{y})}{\partial \mathbf{n}(\mathbf{y})} p(\mathbf{y}) ds(\mathbf{y}) + G(\mathbf{x}_{obs},\mathbf{x}_{src}), \quad \mathbf{x}_{obs} \in \Omega. \tag{4}$$

Our implementation is specifically categorized as the conventional boundary integration method (CBIM), in contrast to a more sophisticated model such as the Burton-Miller method [**?**]. The CBIM often suffers from errors in the frequency where the complementary region of the media $\bar{\Omega} = \{\mathbf{x} \in \mathbb{R}^3 | \mathbf{x} \notin \Omega\}$ has a fictitious resonance mode. In our simulation the complementary region $\bar{\Omega}$ is the solid region of the musical instrument. Since our complementary region $\bar{\Omega}$ is small compared to the cavity, the fictitious resonance mode is much higher compared to the fundamental cavity resonance frequency, and thus CBIM is adequate.

The off-diagonal (i, j)-entry of the resulting coefficient matrix A_{ij} is approximately written as:

$$A_{ij} \simeq \left[\frac{\mathbf{r}_{ij} \cdot \mathbf{n}_i}{4\pi r_{ij}^3} \Delta_j\right] \underbrace{\exp(+ikr_{ij}) \left(1 - ikr_{ij}\right)}_{g(\gamma)}, \tag{5}$$

where $\mathbf{r}_{ij}$ is a vector between i- and j-vertices, $r_{ij} = ||\mathbf{r}_{ij}||$, the $\mathbf{n}_i$ is the unit normal vector, Ω_i is the solid angle at the i-vertex, and Δ_j is one third of the area of triangles around j-vertex. Notice the nonlinearity of the coefficient matrix with respect to wavenumber k (see Sec. 5.1). Furthermore, the nonlinear dependent part $g(\gamma)$ is a function of $\gamma = kr_{ij}$ and if it is small, the linear approximation over the wavenumber is reasonable (see Sec. 6.2). Finally, the entry is invariant under the scaling geometry with s and scaling the wave number with $1/s$ i.e., $r_{ij} \to sr_{ij}$ and $k \to k/s$ (see Sec. 8).