1 FMM 2D

$$m\phi(\mathbf{z},\mathbf{z}_0) = m\log|\mathbf{z}_0-\mathbf{z}| = m\log|\mathbf{z}| - m\sum_{k=1}^{\infty}\frac{(\mathbf{z}_0/\mathbf{z})^k}{k} \tag{1}$$

$$\begin{aligned}
& m\log|(\mathbf{z}_0+\epsilon\mathbf{v}_0)-\mathbf{z}| - m\log|(\mathbf{z}_0-\epsilon\mathbf{v}_0)-\mathbf{z}| && (2)\\
= \; & m\log|\mathbf{z}| - m\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{\mathbf{z}_0+\epsilon\mathbf{v}_0}{\mathbf{z}}\right)^k - m\log|\mathbf{z}| + m\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{\mathbf{z}_0-\epsilon\mathbf{v}_0}{\mathbf{z}}\right)^k && (3)\\
= \; & -m\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{\mathbf{z}_0+\epsilon\mathbf{v}_0}{\mathbf{z}}\right)^k + m\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{\mathbf{z}_0-\epsilon\mathbf{v}_0}{\mathbf{z}}\right)^k && (4)\\
\simeq \; & -2m\sum_{k=1}^{\infty}\frac{1}{k}\frac{k\epsilon\mathbf{z}_0^{k-1}\mathbf{v}_0}{\mathbf{z}^k} && (5)
\end{aligned}$$

$$m\{\mathbf{v}_0\cdot\nabla_{\mathbf{z}_0}\phi(\mathbf{z},\mathbf{z}_0)\} = -m\sum_{k=1}^{\infty}\frac{\mathbf{z}_0^{k-1}\mathbf{v}_0}{\mathbf{z}^k} = -m\left[\sum_{k=1}^{\infty}\left(\frac{\mathbf{z}_0}{\mathbf{z}}\right)^{k-1}\right]\frac{\mathbf{v}_0}{\mathbf{z}} \tag{6}$$

2 Setting

on the boundary $\mathbf{y}\in S$:

$$\mathbf{v}_{\mathbf{y}} + \mathbf{v}^{\infty} = 0 \Leftrightarrow \mathbf{v}_{\mathbf{y}} = -\mathbf{v}^{\infty} \tag{7}$$

3 Helmholtz Decomposition

$$\mathbf{v} = \nabla\phi + \nabla\times\mathbf{A} \tag{8}$$

$$\begin{aligned}
\phi_{\mathbf{x}} &= -\frac{1}{4\pi}\int_V \frac{\nabla_{\mathbf{y}}\cdot\mathbf{v}_{\mathbf{y}}}{||\mathbf{x}-\mathbf{y}||}dV + \frac{1}{4\pi}\int_S \frac{\mathbf{n}_{\mathbf{y}}\cdot\mathbf{v}_{\mathbf{y}}}{||\mathbf{x}-\mathbf{y}||}dS && (9)\\
&= -\frac{1}{4\pi}\int_S \frac{\mathbf{n}_{\mathbf{y}}\cdot\mathbf{v}^{\infty}}{||\mathbf{x}-\mathbf{y}||}dS && (10)\\
&= -\int_S G(\mathbf{x},\mathbf{y})(\mathbf{n}_{\mathbf{y}}\cdot\mathbf{v}^{\infty})dS && (11)
\end{aligned}$$

$$A_{\mathbf{x}} = -\frac{1}{4\pi}\int_V \frac{\nabla_{\mathbf{y}} \times \mathbf{v}_{\mathbf{y}}}{||\mathbf{x}-\mathbf{y}||}dV + \frac{1}{4\pi}\int_S \frac{\mathbf{n}_{\mathbf{y}} \times \mathbf{v}_{\mathbf{y}}}{||\mathbf{x}-\mathbf{y}||}dS \tag{12}$$

$$= -\frac{1}{4\pi}\int_V \frac{\omega_{\mathbf{y}}}{||\mathbf{x}-\mathbf{y}||}dV - \frac{1}{4\pi}\int_S \frac{\mathbf{n}_{\mathbf{y}} \times \mathbf{v}^{\infty}}{||\mathbf{x}-\mathbf{y}||}dS \tag{13}$$

$$= -\int_V G(\mathbf{x},\mathbf{y})(\omega_{\mathbf{y}})dV - \int_S G(\mathbf{x},\mathbf{y})(\mathbf{n}_{\mathbf{y}} \times \mathbf{v}^{\infty})dS \tag{14}$$

4 Boundary Element Method

$$\nabla \cdot (\phi \nabla \psi) = \phi \Delta \psi + \nabla \phi \cdot \nabla \psi \tag{15}$$

$$\int_V \nabla \cdot (\phi \nabla \psi) dv = \int_V \phi \Delta \psi + \nabla \phi \cdot \nabla \psi dv \tag{16}$$

$$\Leftrightarrow \int_V \nabla \cdot (\phi \nabla \psi) dv = \int_S \phi \frac{\partial \psi}{\partial \mathbf{n}} ds \tag{17}$$

$$\Leftrightarrow \int_S \phi \frac{\partial \psi}{\partial \mathbf{n}} ds = \int_V \phi \Delta \psi + \nabla \phi \cdot \nabla \psi dv \tag{18}$$

Green's first theorem

$$\int_S \phi \frac{\partial \psi}{\partial \mathbf{n}} ds = \int_V \phi \Delta \psi + \nabla \phi \cdot \nabla \psi dv \tag{19}$$

Green's second theorem from the Green's first theorem, we can simply derive following equation by substituting the equation from the equation swapping ψ and ϕ.

$$\int_S \left(\phi \frac{\partial \psi}{\partial \mathbf{n}} - \psi \frac{\partial \phi}{\partial \mathbf{n}} \right) ds = \int_V [\phi \Delta \psi - \psi \Delta \phi] \, dv \tag{20}$$

4.1 Kernel function in 3D space

Fundamental solution for Laplace equation in 3D space:

$$G(\mathbf{x},\mathbf{y}) = G(\mathbf{y},\mathbf{x}) = \frac{1}{4\pi||\mathbf{x}-\mathbf{y}||} \tag{21}$$

$\mathbf{r} = \mathbf{x} - \mathbf{y}$. $R = |\mathbf{r}|$

$$\Delta_{\mathbf{x}} G(\mathbf{x},\mathbf{y}) = -\delta(\mathbf{x}-\mathbf{y}), \tag{22}$$

where δ is Dirac delta function.

$$\Delta_{\mathbf{y}} G(\mathbf{x}, \mathbf{y}) = -\delta(\mathbf{x} - \mathbf{y}), \tag{23}$$

where δ is Dirac delta function.

$$\begin{aligned}
\frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial \mathbf{n}_{\mathbf{y}}} &= \frac{\partial}{\partial \mathbf{n}_{\mathbf{y}}} \left(\frac{1}{4\pi |\mathbf{x} - \mathbf{y}|} \right) && (24) \\
&= \frac{\partial}{\partial R} \left(\frac{1}{4\pi R} \right) \frac{\partial R}{\partial \mathbf{n}_{\mathbf{y}}}, \quad (R = |\mathbf{x} - \mathbf{y}|) && (25) \\
&= \frac{-1}{4\pi R^2} \frac{\partial R}{\partial \mathbf{n}_{\mathbf{y}}}, && (26) \\
&= \frac{(\mathbf{x} - \mathbf{y}) \cdot \mathbf{n}}{4\pi |\mathbf{x} - \mathbf{y}|^3} = \frac{\mathbf{r} \cdot \mathbf{n}}{4\pi |\mathbf{r}|^3} && (27)
\end{aligned}$$

Note that the derivative of R in the direction of $\mathbf{n}$ is computed as:

$$\begin{aligned}
\frac{\partial R}{\partial \mathbf{n}_{\mathbf{y}}} &= \frac{\partial R}{\partial \mathbf{y}} \cdot \mathbf{n}_{\mathbf{y}} && (28) \\
&= \frac{\partial \{(\mathbf{x} - \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y})\}^{1/2}}{\partial \mathbf{y}} \cdot \mathbf{n}_{\mathbf{y}} && (29) \\
&= \frac{1}{2} \{(\mathbf{x} - \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y})\}^{-1/2} \frac{\partial (\mathbf{x} - \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y})}{\partial \mathbf{y}} \cdot \mathbf{n}_{\mathbf{y}} && (30) \\
&= \frac{1}{2R} (-2)(\mathbf{x} - \mathbf{y}) \cdot \mathbf{n}_{\mathbf{y}} && (31) \\
&= -\frac{(\mathbf{x} - \mathbf{y}) \cdot \mathbf{n}_{\mathbf{y}}}{R} = -\frac{\mathbf{r} \cdot \mathbf{n}_{\mathbf{y}}}{R} && (32)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial R}{\partial \mathbf{x}} &= \frac{\partial \{(\mathbf{x} - \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y})\}^{1/2}}{\partial \mathbf{x}} && (33) \\
&= \frac{1}{2} \{(\mathbf{x} - \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y})\}^{-1/2} \frac{\partial (\mathbf{x} - \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y})}{\partial \mathbf{x}} && (34) \\
&= \frac{1}{2R} 2(\mathbf{x} - \mathbf{y}) && (35) \\
&= \frac{\mathbf{x} - \mathbf{y}}{R} = \frac{\mathbf{r}}{R} && (36)
\end{aligned}$$

$$\frac{\partial R}{\partial \mathbf{y}} = -\frac{\mathbf{x} - \mathbf{y}}{R} = -\frac{\mathbf{r}}{R} \tag{37}$$

$$\nabla_x G(\mathbf{x}, \mathbf{y}) = \frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial \mathbf{x}} \quad (38)$$

$$= \frac{\partial}{\partial \mathbf{x}} \left(\frac{1}{4\pi |\mathbf{x} - \mathbf{y}|} \right) \quad (39)$$

$$= \frac{\partial}{\partial R} \left(\frac{1}{4\pi R} \right) \frac{\partial R}{\partial \mathbf{x}}, \quad (R = |\mathbf{x} - \mathbf{y}|) \quad (40)$$

$$= \left(\frac{-1}{4\pi R^2} \right) \frac{\mathbf{x} - \mathbf{y}}{R}, \quad (41)$$

$$= -\frac{\mathbf{x} - \mathbf{y}}{4\pi |\mathbf{x} - \mathbf{y}|^3} = -\frac{\mathbf{r}}{4\pi R^3} \quad (42)$$

$$\nabla_y G(\mathbf{x}, \mathbf{y}) = \frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial \mathbf{y}} \quad (43)$$

$$= \frac{\mathbf{x} - \mathbf{y}}{4\pi |\mathbf{x} - \mathbf{y}|^3} = \frac{\mathbf{r}}{4\pi R^3} \quad (44)$$

$$\frac{\partial^2 G(\mathbf{x}, \mathbf{y})}{\partial \mathbf{n}_y \partial \mathbf{x}} = \frac{\partial}{\partial \mathbf{x}} \frac{-(\mathbf{x} - \mathbf{y}) \cdot \mathbf{n}}{4\pi |\mathbf{x} - \mathbf{y}|^3} \quad (45)$$

$$= \frac{\mathbf{n}_y}{4\pi |\mathbf{x} - \mathbf{y}|^3} - \{3(\mathbf{x} - \mathbf{y}) \cdot \mathbf{n}_y\} \frac{\mathbf{x} - \mathbf{y}}{4\pi |\mathbf{x} - \mathbf{y}|^5} \quad (46)$$

$$\frac{\partial^2 G(\mathbf{x}, \mathbf{y})}{\partial \mathbf{x} \partial \mathbf{y}} = \frac{\partial}{\partial \mathbf{x}} \frac{(\mathbf{x} - \mathbf{y})}{4\pi |\mathbf{x} - \mathbf{y}|^3} \quad (47)$$

$$= \frac{\mathbf{I}}{4\pi |\mathbf{x} - \mathbf{y}|^3} - 3 \frac{(\mathbf{x} - \mathbf{y}) \otimes (\mathbf{x} - \mathbf{y})}{4\pi |\mathbf{x} - \mathbf{y}|^5} \quad (48)$$

4.2 Kernel function in 2D space

Fundamental solution for Laplace equation in 2D space:

$$G(\mathbf{x}, \mathbf{y}) = G(\mathbf{y}, \mathbf{x}) = -\frac{1}{2\pi} \log \|\mathbf{x} - \mathbf{y}\| = \frac{1}{2\pi} \log R \quad (49)$$

$\mathbf{r} = \mathbf{x} - \mathbf{y}$. $R = |\mathbf{r}|$

$$\nabla_x G(\mathbf{x}, \mathbf{y}) = -\frac{\mathbf{x} - \mathbf{y}}{2\pi |\mathbf{x} - \mathbf{y}|^2} = -\frac{\mathbf{r}}{2\pi R^2} \quad (50)$$

4.3 BEM

$$\int_S \left(\phi \frac{\partial G(\mathbf{x},\mathbf{y})}{\partial \mathbf{n}_\mathbf{y}} - G(\mathbf{x},\mathbf{y}) \frac{\partial \phi}{\partial \mathbf{n}_\mathbf{y}} \right) ds_\mathbf{y} = \int_\Omega \left[\phi \Delta_\mathbf{y} G(\mathbf{x},\mathbf{y}) - G(\mathbf{x},\mathbf{y}) \Delta_\mathbf{y} \phi \right] dv_\mathbf{y} \tag{51}$$

$$\frac{\Omega(\mathbf{x})}{4\pi} \phi_\mathbf{x} = - \int_S \left(\phi_\mathbf{y} \frac{\partial G(\mathbf{x},\mathbf{y})}{\partial \mathbf{n}_\mathbf{y}} - G(\mathbf{x},\mathbf{y}) \frac{\partial \phi_\mathbf{y}}{\partial \mathbf{n}_\mathbf{y}} \right) ds_\mathbf{y} - \int_V G(\mathbf{x},\mathbf{y}) \Delta_\mathbf{y} \phi_\mathbf{y} dv_\mathbf{y} \tag{52}$$

5 BEM vortex

5.1 Identities

$$\nabla \times (\mathbf{g} \times \mathbf{v}) = (\mathbf{v} \cdot \nabla)\mathbf{g} - (\mathbf{g} \cdot \nabla)\mathbf{v} + \mathbf{g}(\nabla \cdot \mathbf{v}) - \mathbf{v}(\nabla \cdot \mathbf{g}) \tag{53}$$

$$\nabla(\mathbf{g} \cdot \mathbf{v}) = (\mathbf{g} \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{g} + \mathbf{g} \times (\nabla \times \mathbf{v}) + \mathbf{v} \times (\nabla \times \mathbf{g}) \tag{54}$$

$$\nabla \times (\mathbf{g} \times \mathbf{v}) + \nabla(\mathbf{g} \cdot \mathbf{v}) = 2(\mathbf{v} \cdot \nabla)\mathbf{g} + \mathbf{g}(\nabla \cdot \mathbf{v}) - \mathbf{v}(\nabla \cdot \mathbf{g}) \tag{55}$$
$$+ \mathbf{g} \times (\nabla \times \mathbf{v}) + \mathbf{v} \times (\nabla \times \mathbf{g}) \tag{56}$$

$$\int_V \nabla \times (\mathbf{g} \times \mathbf{v}) + \nabla(\mathbf{g} \cdot \mathbf{v}) dv = \int_S \mathbf{n} \times (\mathbf{g} \times \mathbf{v}) + \mathbf{n}(\mathbf{g} \cdot \mathbf{v}) dv \tag{57}$$

$$\mathbf{n} \times (\mathbf{g} \times \mathbf{v}) = (\mathbf{n} \cdot \mathbf{v})\mathbf{g} - (\mathbf{n} \cdot \mathbf{g})\mathbf{v} \tag{58}$$
$$(\mathbf{g} \cdot \mathbf{v})\mathbf{n} = \mathbf{g} \times (\mathbf{n} \times \mathbf{v}) + (\mathbf{n} \cdot \mathbf{g})\mathbf{v} \tag{59}$$

adding to equations becomes

$$\mathbf{n} \times (\mathbf{g} \times \mathbf{v}) + (\mathbf{g} \cdot \mathbf{v})\mathbf{n} = (\mathbf{n} \cdot \mathbf{v})\mathbf{g} + \mathbf{g} \times (\mathbf{n} \times \mathbf{v}) \tag{60}$$

$$\int_S (\mathbf{n} \cdot \mathbf{v})\mathbf{g} + \mathbf{g} \times (\mathbf{n} \times \mathbf{v}) ds \tag{61}$$
$$= \int_V 2(\mathbf{v} \cdot \nabla)\mathbf{g} + \mathbf{g}(\nabla \cdot \mathbf{v}) - \mathbf{v}(\nabla \cdot \mathbf{g}) + \mathbf{g} \times (\nabla \times \mathbf{v}) + \mathbf{v} \times (\nabla \times \mathbf{g}) dv \tag{62}$$

Divergence theorem

$$\int_V \boldsymbol{\nabla} \cdot (g_i \mathbf{v}) dv = \int_V g_i (\boldsymbol{\nabla} \cdot \mathbf{v}) + \mathbf{v} \cdot (\boldsymbol{\nabla} g_i) dv = \int_S g_i (\mathbf{v} \cdot \mathbf{n}) ds \tag{63}$$

$$\Leftrightarrow \int_V \mathbf{g}(\boldsymbol{\nabla} \cdot \mathbf{v}) + (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{g} dv = \int_S \mathbf{g}(\mathbf{v} \cdot \mathbf{n}) ds \tag{64}$$

$$\int_S (\mathbf{n} \cdot \mathbf{v})\mathbf{g} - \mathbf{g} \times (\mathbf{n} \times \mathbf{v}) ds \tag{65}$$

$$= \int_V \mathbf{g}(\boldsymbol{\nabla} \cdot \mathbf{v}) + \mathbf{v}(\boldsymbol{\nabla} \cdot \mathbf{g}) - \mathbf{g} \times (\boldsymbol{\nabla} \times \mathbf{v}) - \mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{g}) dv \tag{66}$$

$$\int_V \mathbf{v}(\nabla \cdot \mathbf{g}) dv = \beta \mathbf{v} \tag{67}$$

$$\int_V \mathbf{v} \times (\nabla \times \mathbf{g}) dv = 0 \tag{68}$$

$$\beta \mathbf{v} = \int_S (\mathbf{n} \cdot \mathbf{v})\mathbf{g} - \mathbf{g} \times (\mathbf{n} \times \mathbf{v}) ds - \int_V (\boldsymbol{\nabla} \cdot \mathbf{v})\mathbf{g} - \mathbf{g} \times (\boldsymbol{\nabla} \times \mathbf{v}) dv \tag{69}$$

when we assume the fluid is incompressible, this becomes

$$\beta \mathbf{v} = \int_S (\mathbf{n} \cdot \mathbf{v})\mathbf{g} - \mathbf{g} \times (\mathbf{n} \times \mathbf{v}) ds + \int_V \mathbf{g} \times \omega dv \tag{70}$$

5.2 Stagnation enthalpy

$$H = \frac{p - p^{\infty}}{\rho} + \frac{1}{2}\left(|\mathbf{v}^{\infty} + \mathbf{v}|^2 - |\mathbf{v}^{\infty}|^2\right) \tag{71}$$

$$\beta_{\mathbf{x}} H_{\mathbf{x}} - \int_S H_{\mathbf{y}} \frac{\partial G_{\mathbf{xy}}}{\partial \mathbf{n}_{\mathbf{y}}} ds_{\mathbf{y}} \tag{72}$$

$$= \int_S \mathbf{n}_{\mathbf{y}} \cdot \frac{\partial(\mathbf{v}_{\mathbf{y}}^{\infty} + \mathbf{v}_{\mathbf{y}})}{\partial t} G_{\mathbf{xy}} + \nu \nabla_{\mathbf{y}} G_{\mathbf{xy}} \cdot (\mathbf{n}_{\mathbf{y}} \times \omega_{\mathbf{y}}) ds_{\mathbf{y}} \tag{73}$$

$$- \int_V \nabla_{\mathbf{y}} G_{\mathbf{xy}} \cdot \left\{(\mathbf{v}_{\mathbf{y}}^{\infty} + \mathbf{v}_{\mathbf{y}}) \times \omega_{\mathbf{y}}\right\} dv_{\mathbf{y}} \tag{74}$$

5.3 NS-equation

$$\boldsymbol{\omega} = \nabla \times \mathbf{v} \tag{75}$$

$$\nabla \cdot \mathbf{v} = 0 \tag{76}$$

$$\nabla \cdot \nabla \mathbf{v} + \nabla \times \boldsymbol{\omega} = 0 \tag{77}$$

$$\Delta_{\mathbf{y}} \mathbf{v} = -\nabla_{\mathbf{y}} \times \boldsymbol{\omega}_{\mathbf{y}} \tag{78}$$

The boundary integral representation for the the velocity becomes:

$$\frac{\Omega_{\mathbf{x}}}{4\pi} \mathbf{v}_{\mathbf{x}} = \int_S \left[\mathbf{v}_{\mathbf{y}} \frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial \mathbf{n}_{\mathbf{y}}} - G(\mathbf{x}, \mathbf{y}) \frac{\partial \mathbf{v}_{\mathbf{y}}}{\partial \mathbf{n}_{\mathbf{y}}} \right] ds_{\mathbf{y}} - \int_V (\nabla_{\mathbf{y}} \times \boldsymbol{\omega}_{\mathbf{y}}) G(\mathbf{x}, \mathbf{y}) dv_{\mathbf{y}} \tag{79}$$

$$(\nabla_{\mathbf{y}} \times \boldsymbol{\omega}_{\mathbf{y}}) G(\mathbf{x}, \mathbf{y}) = \nabla_{\mathbf{y}} \times (\boldsymbol{\omega}_{\mathbf{y}} G(\mathbf{x}, \mathbf{y})) + \boldsymbol{\omega}_{\mathbf{y}} \times \nabla_{\mathbf{y}} G(\mathbf{x}, \mathbf{y}) \tag{80}$$

$$\int_V (\nabla_{\mathbf{y}} \times \boldsymbol{\omega}_{\mathbf{y}}) G(\mathbf{x}, \mathbf{y}) dv_{\mathbf{y}} = \int_V \nabla_{\mathbf{x}} \times (\boldsymbol{\omega}_{\mathbf{y}} G(\mathbf{x}, \mathbf{y})) dv_{\mathbf{y}} + \int_V \boldsymbol{\omega}_{\mathbf{y}} \times \nabla_{\mathbf{y}} G(\mathbf{x}, \mathbf{y}) dv_{\mathbf{y}} \tag{81}$$

$$= \int_S \mathbf{n}_{\mathbf{y}} \times (\boldsymbol{\omega}_{\mathbf{y}} G(\mathbf{x}, \mathbf{y})) ds_{\mathbf{y}} + \int_V \boldsymbol{\omega}_{\mathbf{y}} \times \nabla_{\mathbf{y}} G(\mathbf{x}, \mathbf{y}) dv_{\mathbf{y}} \tag{82}$$

$$\frac{\Omega_{\mathbf{x}}}{4\pi} \mathbf{v}_{\mathbf{x}} = \int_S \left[\mathbf{v}_{\mathbf{y}} \frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial \mathbf{n}_{\mathbf{y}}} - G(\mathbf{x}, \mathbf{y}) \frac{\partial \mathbf{v}_{\mathbf{y}}}{\partial \mathbf{n}_{\mathbf{y}}} \right] ds_{\mathbf{y}}$$

$$- \int_S (\mathbf{n}_{\mathbf{y}} \times \boldsymbol{\omega}_{\mathbf{y}}) G(\mathbf{x}, \mathbf{y})) ds_{\mathbf{y}} - \int_V \boldsymbol{\omega}_{\mathbf{y}} \times \nabla_{\mathbf{x}} G(\mathbf{x}, \mathbf{y}) dv_{\mathbf{y}} \tag{83}$$

$$= \int_S \mathbf{v}_{\mathbf{y}} \frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial \mathbf{n}_{\mathbf{y}}} - G(\mathbf{x}, \mathbf{y}) \left\{ \frac{\partial \mathbf{v}_{\mathbf{y}}}{\partial \mathbf{n}_{\mathbf{y}}} + (\mathbf{n}_{\mathbf{y}} \times \boldsymbol{\omega}_{\mathbf{y}}) \right\} ds_{\mathbf{y}}$$

$$- \int_V \boldsymbol{\omega}_{\mathbf{y}} \times \nabla_{\mathbf{y}} G(\mathbf{x}, \mathbf{y}) dv_{\mathbf{y}} \tag{84}$$

$$
\begin{aligned}
\nabla(\mathbf{n}\cdot\mathbf{v}) &= (\mathbf{n}\cdot\nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{n} + \mathbf{n}\times(\nabla\times\mathbf{v}) + \mathbf{v}\times(\nabla\times\mathbf{n}) && (85)\\
\Leftrightarrow \quad \nabla(\mathbf{n}\cdot\mathbf{v}) &= \frac{\partial\mathbf{v}}{\partial\mathbf{n}} + (\mathbf{v}\cdot\nabla)\mathbf{n} + \mathbf{n}\times\omega + \mathbf{v}\times(\nabla\times\mathbf{n}) && (86)\\
\Leftrightarrow \quad \frac{\partial\mathbf{v}}{\partial\mathbf{n}} + \mathbf{n}\times\omega &= \nabla(\mathbf{n}\cdot\mathbf{v}) - (\mathbf{v}\cdot\nabla)\mathbf{n} - \mathbf{v}\times(\nabla\times\mathbf{n}) && (87)
\end{aligned}
$$

$$
\nabla\{G(\mathbf{n}\cdot\mathbf{v})\} = G\nabla(\mathbf{n}\cdot\mathbf{v}) + (\mathbf{n}\cdot\mathbf{v})\nabla G \qquad (88)
$$

$$
\begin{aligned}
\nabla\times\{G(\mathbf{v}\times\mathbf{n})\} &= G\{\nabla\times(\mathbf{v}\times\mathbf{n})\} + \{\nabla G\}\times(\mathbf{v}\times\mathbf{n}) && (89)\\
&= G\{\mathbf{v}(\nabla\cdot\mathbf{n}) - \mathbf{n}(\nabla\cdot\mathbf{v}) + (\mathbf{n}\cdot\nabla)\mathbf{v} - (\mathbf{v}\cdot\nabla)\mathbf{n}\}\\
&\quad + \{\nabla G\}\times(\mathbf{v}\times\mathbf{n}) && (90)
\end{aligned}
$$

5.4 Vortex Sheet

$$
\begin{aligned}
\frac{\Omega_{\mathbf{x}}}{4\pi}\mathbf{v}_{\mathbf{x}} = &-\int_V \omega_{\mathbf{y}}\times\nabla_{\mathbf{y}}G(\mathbf{x},\mathbf{y})dv_{\mathbf{y}}\\
&-\int_S (\mathbf{n}_{\mathbf{y}}\cdot\mathbf{v}_{\mathbf{y}})\nabla_{\mathbf{y}}G(\mathbf{x},\mathbf{y}) - (\mathbf{n}_{\mathbf{y}}\times\mathbf{v}_{\mathbf{y}})\times\nabla_{\mathbf{y}}G(\mathbf{x},\mathbf{y})ds_{\mathbf{y}} \qquad (91)
\end{aligned}
$$

non-slip condition:

$$
\mathbf{v}_{\mathbf{x}} = \boldsymbol{\gamma}_{\mathbf{x}}\times\mathbf{n}_{\mathbf{x}} \quad \mathbf{x}\in S \qquad (92)
$$

$\mathbf{y}\in S$

$$
\begin{aligned}
\mathbf{n}_{\mathbf{y}}\times\mathbf{v}_{\mathbf{y}} &= \mathbf{n}_{\mathbf{y}}\times(\boldsymbol{\gamma}_{\mathbf{y}}\times\mathbf{n}_{\mathbf{y}}) && (93)\\
&= (\mathbf{n}_{\mathbf{y}}\cdot\mathbf{n}_{\mathbf{y}})\boldsymbol{\gamma}_{\mathbf{y}} - (\mathbf{n}_{\mathbf{y}}\cdot\boldsymbol{\gamma}_{\mathbf{y}})\mathbf{n}_{\mathbf{y}} && (94)\\
&= \boldsymbol{\gamma}_{\mathbf{y}} && (95)
\end{aligned}
$$

$$\frac{\Omega_{\mathbf{x}}}{4\pi}(\mathbf{n}_{\mathbf{x}} \times \mathbf{v}_{\mathbf{x}}) = -\mathbf{n}_{\mathbf{x}} \times \left\{ \int_V \omega_{\mathbf{y}} \times \nabla_{\mathbf{y}} G(\mathbf{x}, \mathbf{y}) dv_{\mathbf{y}} \right\} \tag{96}$$

$$- \mathbf{n}_{\mathbf{x}} \times \left\{ \int_S (\mathbf{n}_{\mathbf{y}} \cdot \mathbf{v}_{\mathbf{y}}) \nabla_{\mathbf{y}} G(\mathbf{x}, \mathbf{y}) ds_{\mathbf{y}} \right\} \tag{97}$$

$$+ \mathbf{n}_{\mathbf{x}} \times \left\{ \int_S (\mathbf{n}_{\mathbf{y}} \times \mathbf{v}_{\mathbf{y}}) \times \nabla_{\mathbf{y}} G(\mathbf{x}, \mathbf{y}) ds_{\mathbf{y}} \right\} \tag{98}$$

$$\Leftrightarrow \frac{\Omega_{\mathbf{x}}}{4\pi}\gamma_{\mathbf{x}} = -\mathbf{n}_{\mathbf{x}} \times \left\{ \int_V \omega_{\mathbf{y}} \times \nabla_{\mathbf{y}} G(\mathbf{x}, \mathbf{y}) dv_{\mathbf{y}} \right\} \tag{99}$$

$$- \mathbf{n}_{\mathbf{x}} \times \left\{ \int_S (\mathbf{n}_{\mathbf{y}} \cdot \mathbf{v}_{\mathbf{y}}) \nabla_{\mathbf{y}} G(\mathbf{x}, \mathbf{y}) ds_{\mathbf{y}} \right\} \tag{100}$$

$$+ \mathbf{n}_{\mathbf{x}} \times \left\{ \int_S \gamma_{\mathbf{y}} \times \nabla_{\mathbf{y}} G(\mathbf{x}, \mathbf{y}) ds_{\mathbf{y}} \right\} \tag{101}$$

$$\tag{102}$$

5.5 Misc.

vorticity transport equation:

$$\frac{\partial \omega}{\partial t} + (\mathbf{v} \cdot \nabla)\omega = (\omega \cdot \nabla)\mathbf{v} + \nu \nabla \cdot \nabla \omega, \tag{103}$$

where $\mathbf{v}$ is a velocity and ω is a vorticity

$$\frac{\Omega(\mathbf{x})}{4\pi}\omega(\mathbf{x}) \tag{104}$$

$$= \int_S \left[\omega_{\mathbf{y}} \frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial \mathbf{n}_{\mathbf{y}}} - G(\mathbf{x}, \mathbf{y}) \frac{\partial \omega_{\mathbf{y}}}{\partial \mathbf{n}_{\mathbf{y}}} \right] ds_{\mathbf{y}} \tag{105}$$

$$+ \frac{1}{\nu} \int_V G(\mathbf{x}, \mathbf{y}) \{ (\mathbf{v} \cdot \nabla)\omega - (\omega \cdot \nabla)\mathbf{v} \} dv_{\mathbf{y}} \tag{106}$$

material derivative:

$$\frac{D\omega}{Dt} = (\omega \cdot \nabla)\mathbf{v} + \nu \nabla \cdot \nabla \omega, \tag{107}$$

$$\mathbf{v} = \mathbf{v}^{\omega} + \mathbf{v}^{\phi} + \mathbf{v}^{\infty} \tag{108}$$

$\mathbf{v}^{\phi} = \nabla \phi$

$$\begin{aligned}\mathbf{v}\cdot\mathbf{n} &= 0 \quad (109)\\ \mathbf{v}^{\phi}\cdot\mathbf{n} &= -(\mathbf{v}^{\omega}+\mathbf{v}^{\infty})\cdot\mathbf{n} \quad (110)\\ \frac{\partial\phi}{\partial\mathbf{n}} &= -(\mathbf{v}^{\omega}+\mathbf{v}^{\infty})\cdot\mathbf{n}=\alpha \quad (111)\end{aligned}$$

$$\nabla\cdot\mathbf{v}^{\phi}=0\Leftrightarrow\Delta\phi=0 \quad (112)$$

6 Potential Flow

velocity: $\mathbf{v}^{\infty}+\mathbf{v}$ and $\mathbf{v}=\nabla\phi$.

no influx:

$$\begin{aligned}&(\mathbf{v}^{\infty}+\mathbf{v})\cdot\mathbf{n}=0 \quad (113)\\ \Leftrightarrow&\mathbf{v}\cdot\mathbf{n}=-\mathbf{v}^{\infty}\cdot\mathbf{n} \quad (114)\\ \Leftrightarrow&\frac{\partial\phi}{\partial\mathbf{n}}=-\mathbf{v}^{\infty}\cdot\mathbf{n} \quad (115)\end{aligned}$$

boundary condition for potential flow.

$$\begin{aligned}\phi_{\mathbf{x}} &= 0,\quad \text{for } |\mathbf{x}|\to\infty \quad (116)\\ \Delta\phi_{\mathbf{x}} &= 0,\quad \text{for } \mathbf{x}\in\Omega \quad (117)\end{aligned}$$

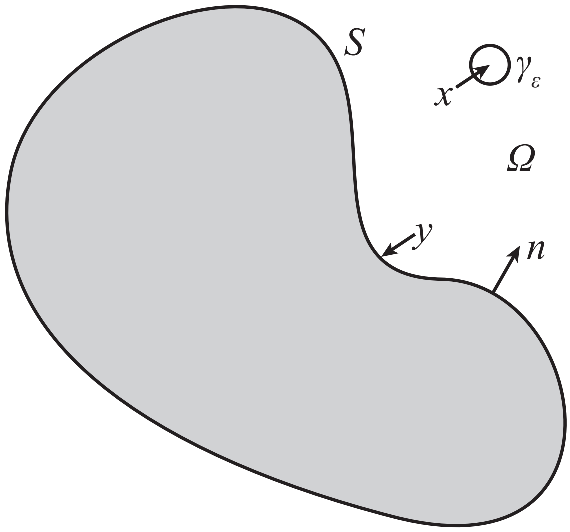
$$\begin{aligned}\frac{\partial\phi_{\mathbf{y}}}{\partial\mathbf{n}_{\mathbf{y}}} &= \alpha_{\mathbf{y}},\quad \text{for } \mathbf{y}\in S \quad (118)\\ & \quad (119)\end{aligned}$$

$$\phi_{\mathbf{x}}=-\int_S\left[\phi_{\mathbf{y}}\frac{\partial G(\mathbf{x},\mathbf{y})}{\partial\mathbf{n}_{\mathbf{y}}}-G(\mathbf{x},\mathbf{y})\frac{\partial\phi_{\mathbf{y}}}{\partial\mathbf{n}_{\mathbf{y}}}\right]ds_{\mathbf{y}},\quad \mathbf{x}\in\Omega \quad (120)$$

$$\frac{\Omega_{\mathbf{x}}}{4\pi}\phi_{\mathbf{x}}=-\int_S\left[\phi_{\mathbf{y}}\frac{\partial G(\mathbf{x},\mathbf{y})}{\partial\mathbf{n}_{\mathbf{y}}}-G(\mathbf{x},\mathbf{y})\frac{\partial\phi_{\mathbf{y}}}{\partial\mathbf{n}_{\mathbf{y}}}\right]ds_{\mathbf{y}},\quad \mathbf{x}\in S \quad (121)$$

$$\frac{\Omega_{\mathbf{x}}}{4\pi}\phi_{\mathbf{x}}+\int_S\phi_{\mathbf{y}}\frac{\partial G(\mathbf{x},\mathbf{y})}{\partial\mathbf{n}_{\mathbf{y}}}ds_{\mathbf{y}}=\int_S G(\mathbf{x},\mathbf{y})\frac{\partial\phi_{\mathbf{y}}}{\partial\mathbf{n}_{\mathbf{y}}}ds_{\mathbf{y}},\quad \mathbf{x}\in S \quad (122)$$

$$\begin{aligned}\mathbf{f}_i=\int_S G(\mathbf{x}_i,\mathbf{y})\alpha_{\mathbf{y}}ds_{\mathbf{y}},\quad \mathbf{x}_i\in S \quad (123)\\ (124)\end{aligned}$$



7 Evaluation of Solution

$$\mathbf{v}^{\phi}_{\mathbf{x}} = \nabla_{\mathbf{x}}\phi \tag{125}$$

$$\nabla_{\mathbf{x}}\phi_{\mathbf{x}} = -\int_S \left[\phi_{\mathbf{y}} \frac{\partial^2 G(\mathbf{x},\mathbf{y})}{\partial \mathbf{n}(\mathbf{y})\partial \mathbf{x}} - \frac{\partial G(\mathbf{x},\mathbf{y})}{\partial \mathbf{x}} \frac{\partial \phi_{\mathbf{y}}}{\partial \mathbf{n}_{\mathbf{y}}}\right] ds_{\mathbf{y}}, \quad \mathbf{x} \in \Omega \tag{126}$$

$$= -\int_S \left[\phi_{\mathbf{y}} \frac{\partial^2 G(\mathbf{x},\mathbf{y})}{\partial \mathbf{n}_{\mathbf{y}}\partial \mathbf{x}} - \frac{\partial G(\mathbf{x},\mathbf{y})}{\partial \mathbf{x}} \alpha_{\mathbf{y}}\right] ds_{\mathbf{y}}, \quad \mathbf{x} \in \Omega \tag{127}$$

$$\tag{128}$$

7.1 Stabilization

assuming ϕ is distributed spatially around the center of a triangle $\mathbf{y}_e$.

$$\phi_{\mathbf{y}} = \phi_e A f_\sigma(|\mathbf{y} - \mathbf{y}_e|) \tag{129}$$

$$f_\sigma(r) = \frac{3}{4\pi\sigma^3} \exp\left\{-(r/\sigma)^3\right\} \tag{130}$$

where A is area of the triangle, σ is the representative length of an element.

$$\phi_{\mathbf{x}} = -\int_S \left[\phi_{\mathbf{y}} \frac{\partial G(\mathbf{x},\mathbf{y})}{\partial \mathbf{n}_{\mathbf{y}}} - G(\mathbf{x},\mathbf{y}) \frac{\partial \phi_{\mathbf{y}}}{\partial \mathbf{n}_{\mathbf{y}}} \right] ds_{\mathbf{y}} \tag{131}$$

$$= -\sum_e \int_V \left[\phi_{\mathbf{y}} \frac{\partial G(\mathbf{x},\mathbf{y})}{\partial \mathbf{n}_{\mathbf{y}}} - G(\mathbf{x},\mathbf{y}) \frac{\partial \phi_{\mathbf{y}}}{\partial \mathbf{n}_{\mathbf{y}}} \right] dv_{\mathbf{y}} \tag{132}$$

$$\tag{133}$$

$$\frac{\partial}{\partial R} \left[1 - \exp\left\{ -(R/\sigma)^3 \right\} \right] \tag{134}$$

$$= \left[-\exp\left\{ -(R/\sigma)^3 \right\} \right] \frac{\partial}{\partial R} \left\{ -(R/\sigma)^3 \right\} \tag{135}$$

$$= 3R^2/\sigma^3 \exp\left\{ -(R/\sigma)^3 \right\} \tag{136}$$

$$\frac{\partial}{\partial \mathbf{x}} \left[1 - \exp\left\{ -(R/\sigma)^3 \right\} \right] \tag{137}$$

$$= \frac{\partial}{\partial R} \left[1 - \exp\left\{ -(R/\sigma)^3 \right\} \right] \frac{\partial R}{\partial \mathbf{x}} \tag{138}$$

$$= 3R/\sigma^3 \exp\left\{ -(R/\sigma)^3 \right\} \mathbf{r} \tag{139}$$

8 Vortex

Wu and Thompson 1973:

$$\mathbf{v}_{\mathbf{x}} = \mathbf{v}^{\infty} \underbrace{- \int_{\Omega} \boldsymbol{\omega}_{\mathbf{y}} \times \nabla_{\mathbf{x}} G(\mathbf{x},\mathbf{y}) ds_{\mathbf{y}}}_{\mathbf{v}^{\omega}} \tag{140}$$

$$\underbrace{- \int_S (\mathbf{n}_{\mathbf{y}} \cdot \mathbf{v}_{\mathbf{y}}) \nabla_{\mathbf{x}} G(\mathbf{x},\mathbf{y}) ds_{\mathbf{y}}}_{\mathbf{v}^{\phi}} \tag{141}$$

$$+ \int_S (\boldsymbol{\gamma}_{\mathbf{y}} + \mathbf{n}_{\mathbf{y}} \times \mathbf{v}_{\mathbf{y}}) \times \nabla_{\mathbf{x}} G(\mathbf{x},\mathbf{y}) ds_{\mathbf{y}} \tag{142}$$

$$= \mathbf{v}^{\infty} - \nabla_{\mathbf{x}} \times \int_{\Omega} \boldsymbol{\omega}_{\mathbf{y}} G(\mathbf{x},\mathbf{y}) ds_{\mathbf{y}} \tag{143}$$

$$+ \nabla_{\mathbf{x}} \underbrace{\int_S (-\mathbf{n}_{\mathbf{y}} \cdot \mathbf{v}_{\mathbf{y}}) G(\mathbf{x},\mathbf{y}) ds_{\mathbf{y}}}_{\phi} \tag{144}$$

$$+ \nabla_{\mathbf{x}} \times \int_S (\boldsymbol{\gamma}_{\mathbf{y}} + \mathbf{n}_{\mathbf{y}} \times \mathbf{v}_{\mathbf{y}}) G(\mathbf{x},\mathbf{y}) ds_{\mathbf{y}} \tag{145}$$

when $\mathbf{n_y} \cdot \mathbf{v_y} = 0, \mathbf{n_y} \times \mathbf{v_y} = 0$

$$\mathbf{v_x} - \int_S \boldsymbol{\gamma}_\mathbf{y} \times \nabla_\mathbf{x} G(\mathbf{x}, \mathbf{y}) ds_\mathbf{y} = \mathbf{v}^\infty - \int_\Omega \omega_\mathbf{y} \times \nabla_\mathbf{x} G(\mathbf{x}, \mathbf{y}) ds_\mathbf{y} \tag{146}$$

at the boundary, we have

$$\frac{\Omega(\mathbf{x})}{4\pi}\mathbf{v_x} - \int_S \boldsymbol{\gamma}_\mathbf{y} \times \nabla_\mathbf{x} G(\mathbf{x}, \mathbf{y}) ds_\mathbf{y} = \mathbf{v}^\infty - \int_\Omega \omega_\mathbf{y} \times \nabla_\mathbf{x} G(\mathbf{x}, \mathbf{y}) ds_\mathbf{y} \tag{147}$$

$$\mathbf{v} = (\mathbf{v} \cdot \mathbf{n})\mathbf{n} + (\mathbf{n} \times \mathbf{v}) \times \mathbf{n} \tag{148}$$

$$\frac{\Omega(\mathbf{x})}{4\pi}(\mathbf{v_x} \cdot \mathbf{n_x}) = \mathbf{U}_\infty \cdot \mathbf{n_x} - \int_\Omega \mathbf{n_x} \cdot \{\omega_\mathbf{y} \times \nabla_\mathbf{x} G(\mathbf{x}, \mathbf{y})\} ds_\mathbf{y} \tag{149}$$

$$- \int_S (\mathbf{n_y} \cdot \mathbf{v_y}) \{\mathbf{n_x} \cdot \nabla_\mathbf{x} G(\mathbf{x}, \mathbf{y})\} ds_\mathbf{y} \tag{150}$$

$$+ \int_S \mathbf{n_x} \cdot \left\{(\boldsymbol{\gamma}_\mathbf{y} + \mathbf{n_y} \times \mathbf{v_y}) \times \nabla_\mathbf{x} G(\mathbf{x}, \mathbf{y})\right\} ds_\mathbf{y} \tag{151}$$

8.1 Vortex Blob

each vortex blob has circulation $\boldsymbol{\Gamma}_k$, location $\mathbf{y}_k$, and size σ_k.

$$\omega_\mathbf{y} = \sum_{k=1} \boldsymbol{\Gamma}_k g(\mathbf{y} - \mathbf{y}_k, \sigma_k) \tag{152}$$

The g is a Gaussian distribution

$$g(\mathbf{y}, \boldsymbol{\Sigma}) = \frac{1}{\sqrt{(2\pi)^3 |\boldsymbol{\Sigma}|}} \exp\left(-\frac{1}{2}\mathbf{y}^T \Sigma^{-1} \mathbf{y}\right) \tag{153}$$

The g is a isotropic Gaussian distribution

$$g(\mathbf{y}, \sigma) = \frac{1}{\sqrt{8\pi^3 \sigma^2}} \exp\left(-\frac{|\mathbf{y}|^2}{2\sigma^2}\right) \tag{154}$$

$$\tag{155}$$

Rankine vortex particle

$$g(\mathbf{y}, \sigma) = \begin{cases} \frac{3}{4\pi\sigma^3} & (r \leq \sigma) \\ 0 & (r > \sigma) \end{cases} \tag{156}$$

Another modeling

$$g(\mathbf{y}, \sigma) = \frac{3}{4\pi\sigma^3} \exp\left(-\frac{|\mathbf{y}|^3}{\sigma^3}\right) \tag{157}$$

$$\begin{aligned}
\mathbf{v}_{\mathbf{x}}^{\omega} &= -\int_V \boldsymbol{\omega}_{\mathbf{y}} \times \nabla_{\mathbf{y}} G(\mathbf{x}, \mathbf{y}) dv_{\mathbf{y}} && (158) \\
&= -\int_V \left\{ \sum_{k=1} \boldsymbol{\Gamma}_k g(\mathbf{y} - \mathbf{y}_k, \sigma) \right\} \times \left\{ -\frac{\mathbf{x} - \mathbf{y}}{4\pi|\mathbf{x} - \mathbf{y}|^3} \right\} dv_{\mathbf{y}} && (159) \\
&= \sum_{k=1} \boldsymbol{\Gamma}_k \times \left\{ \int_V g(\mathbf{y} - \mathbf{y}_k, \sigma) \frac{\mathbf{x} - \mathbf{y}}{4\pi|\mathbf{x} - \mathbf{y}|^3} dv_{\mathbf{y}} \right\} && (160) \\
&= \sum_{k=1} \boldsymbol{\Gamma}_k \times \left\{ \frac{\mathbf{x} - \mathbf{y}_k}{4\pi|\mathbf{x} - \mathbf{y}_k|^3} \int_0^{|\mathbf{x} - \mathbf{y}_k|/\sigma} g(r, \sigma) r^2 dr \right\} && (161) \\
&= \sum_{k=1} \boldsymbol{\Gamma}_k \times \left[\frac{\mathbf{x} - \mathbf{y}_k}{4\pi|\mathbf{x} - \mathbf{y}_k|^3} \left\{ 1 - \exp\left(-\frac{|\mathbf{x} - \mathbf{y}_k|^3}{\sigma^3} \right) \right\} \right] && (162) \\
&= \sum_{k=1} \boldsymbol{\Gamma}_k \times \nabla_{\mathbf{x}} G(\mathbf{x}, \mathbf{y}_k) f\left(\frac{|\mathbf{x} - \mathbf{y}_k|}{\sigma} \right) && (163)
\end{aligned}$$