Implementation Note for MITC3

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1 Introduction

In this document, we explain the detailed implementation of the MITC3 shell element presented by Lee et al. **?**.

Here, we assume the "plate bending" situation where in undeformed state a flat plate is placed on the XY-plane with Z-axis director vectors. In deformed sate, the plate is deflected in the Z-axis direction and the director vector is rotated with an axis that is orthogonal to the Z-axis.

2 Undeformed Position

In general case, the position inside the MITC shell can be written using the natural coordinate (r, s, t) as:

$$\begin{pmatrix} X(r,s,t) \\ Y(r,s,t) \\ Z(r,s,t) \end{pmatrix} = \sum_{i=1}^{3} L^i(r,s) \begin{pmatrix} X^i \\ Y^i \\ Z^i \end{pmatrix} + \frac{at}{2} \sum_{i=1}^{3} L^i(r,s) \begin{pmatrix} N_x^i \\ N_y^i \\ N_z^i \end{pmatrix}, \quad (1)$$

where the a is the thickness of the plate. L^i is the shape function of the triangle element $L^0 = 1 - r - s, L^1 = r, L^2 = s$.

In the plate bending problem setting, the position can be written as:

$$\begin{pmatrix} X(r,s,t) \\ Y(r,s,t) \\ Z(r,s,t) \end{pmatrix} = \sum_{i=1}^{3} L^i(r,s) \begin{pmatrix} X^i \\ Y^i \\ 0 \end{pmatrix} + \frac{at}{2} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad (2)$$

The covariant bases of the embedded coordinates are computed as:

$$\mathbf{G}_r = \begin{pmatrix} X^1 - X^0 \\ Y^1 - Y^0 \\ 0 \end{pmatrix}, \mathbf{G}_s = \begin{pmatrix} X^2 - X^0 \\ Y^2 - Y^0 \\ 0 \end{pmatrix}, \mathbf{G}_t = \begin{pmatrix} 0 \\ 0 \\ a/2 \end{pmatrix}, \quad (3)$$

The contravariant bases of the embedded coordinate are computed as:

$$\mathbf{G}^r = \frac{\mathbf{G}_s \times \mathbf{G}_t}{\mathbf{G}_r \cdot (\mathbf{G}_s \times \mathbf{G}_t)}, \tag{4}$$

$$\mathbf{G}^s = \frac{\mathbf{G}_t \times \mathbf{G}_r}{\mathbf{G}_s \cdot (\mathbf{G}_t \times \mathbf{G}_r)}, \tag{5}$$

$$\mathbf{G}^t = \frac{\mathbf{G}_r \times \mathbf{G}_s}{\mathbf{G}_t \cdot (\mathbf{G}_r \times \mathbf{G}_s)} \tag{6}$$

3 Displacement

we assume the plate bending situation where the displacement at the node has only z component. The point after displacement can be written as:

$$\begin{pmatrix} p_x(r,s,t) \\ p_y(r,s,t) \\ p_z(r,s,t) \end{pmatrix} = \sum_{i=1}^{3} L^i(r,s) \begin{pmatrix} X^i \\ Y^i \\ u_z^i \end{pmatrix} + \frac{at}{2} \sum_{i=1}^{3} L^i(r,s)(I + \tilde{\mathbf{v}}^i) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{7}$$

$\tilde{\mathbf{v}}$ stands for the 3x3 skew-symmetric matrix from the axial vector $\mathbf{v} = (\alpha, \beta, 0)^T$.

Substituting undeformed position (**??**) from the deformed position (**??**), we have displacement as

$$\mathbf{u} = \begin{pmatrix} u_x(r,s,t) \\ u_y(r,s,t) \\ u_z(r,s,t) \end{pmatrix} = L^i(r,s) \begin{pmatrix} +\beta^i at/2 \\ -\alpha^i at/2 \\ u_z^i \end{pmatrix} \tag{8}$$

4 Linear Strain

The gradient of the deformation can be computed by differentiating $\mathbf{u}$ in (**??**) with the natural coordinate (r, s, t).

$$\frac{\partial \mathbf{u}}{\partial r} = \begin{pmatrix} +(\beta^1 - \beta^0)at/2 \\ -(\alpha^1 - \alpha^0)at/2 \\ u_z^1 - u_z^0 \end{pmatrix} \tag{9}$$

$$\frac{\partial \mathbf{u}}{\partial s} = \begin{pmatrix} +(\beta^2 - \beta^0)at/2 \\ -(\alpha^2 - \alpha^0)at/2 \\ u_z^2 - u_z^0 \end{pmatrix}, \tag{10}$$

$$\frac{\partial \mathbf{u}}{\partial t} = L^i(r,s) \begin{pmatrix} +\beta^i a/2 \\ -\alpha^i a/2 \\ 0 \end{pmatrix} \tag{11}$$

Using the embedded coordinates in (**??**), the coeffieicnts of the linear strain can be written as:

$$
\begin{aligned}
\epsilon_{rr} &= \mathbf{G}_r \cdot \frac{\partial \mathbf{u}}{\partial r} && (12)\\
&= +\frac{at}{2}(X^1 - X^0)\cdot(\beta^1 - \beta^0) - \frac{at}{2}(Y^1 - Y^0)\cdot(\alpha^1 - \alpha^0) && (13)\\
\epsilon_{ss} &= \mathbf{G}_s \cdot \frac{\partial \mathbf{u}}{\partial s} && (14)\\
&= +\frac{at}{2}(X^2 - X^0)\cdot(\beta^2 - \beta^0) - \frac{at}{2}(Y^2 - Y^0)\cdot(\alpha^2 - \alpha^0) && (15)\\
\epsilon_{rs} &= \frac{1}{2}\left(\mathbf{G}_r \cdot \frac{\partial \mathbf{u}}{\partial s} + \mathbf{G}_s \cdot \frac{\partial \mathbf{u}}{\partial r}\right) && (16)\\
&= +\frac{at}{4}(X^1 - X^0)\cdot(\beta^2 - \beta^0) - \frac{at}{4}(Y^1 - Y^0)\cdot(\alpha^2 - \alpha^0)\\
&\quad +\frac{at}{4}(X^2 - X^0)\cdot(\beta^1 - \beta^0) - \frac{at}{4}(Y^2 - Y^0)\cdot(\alpha^1 - \alpha^0) && (17)\\
\epsilon_{tt} &= 0 && (18)\\
\epsilon_{rt} &= \frac{1}{2}\left(\mathbf{G}_t \cdot \frac{\partial \mathbf{u}}{\partial r} + \mathbf{G}_r \cdot \frac{\partial \mathbf{u}}{\partial t}\right) && (19)\\
&= \frac{a}{2}\left\{u_z^1 - u_z^0 + \mathbf{G}_r \cdot \left(\sum_{i=1}^{3} L^i \mathbf{v}^i\right)\right\} && (20)\\
\epsilon_{st} &= \frac{1}{2}\left(\mathbf{G}_t \cdot \frac{\partial \mathbf{u}}{\partial s} + \mathbf{G}_s \cdot \frac{\partial \mathbf{u}}{\partial t}\right) && (21)\\
&= \frac{a}{2}\left\{u_z^2 - u_z^0 + \mathbf{G}_s \cdot \left(\sum_{i=1}^{3} L^i \mathbf{v}^i\right)\right\} && (22)
\end{aligned}
$$

Let us define the trying points A,B,C at the edge of the tiangle such that the barycentric coordinates (L^0, L^1, L^2) are $(0.5, 0.5, 0)$ at A, $(0.5, 0, 0.5)$ at B, $(0, 0.5, 0.5)$ at C. The shear strain at these trying point can be written as:

$$
\begin{aligned}
\epsilon_{rt}^A &= \frac{a}{2}\left\{u_z^1 - u_z^0 + 0.5(X^1 - X^0)(\beta^0 + \beta^1) - 0.5(Y^1 - Y^0)(\alpha^0 + \alpha^1)\right\} && (23)\\
\epsilon_{rt}^C &= \frac{a}{2}\left\{u_z^1 - u_z^0 + 0.5(X^1 - X^0)(\beta^1 + \beta^2) - 0.5(Y^1 - Y^0)(\alpha^1 + \alpha^2)\right\} && (24)\\
\epsilon_{st}^B &= \frac{a}{2}\left\{u_z^2 - u_z^0 + 0.5(X^2 - X^0)(\beta^0 + \beta^1) - 0.5(Y^2 - Y^0)(\alpha^0 + \alpha^1)\right\} && (25)\\
\epsilon_{st}^C &= \frac{a}{2}\left\{u_z^2 - u_z^0 + 0.5(X^2 - X^0)(\beta^1 + \beta^2) - 0.5(Y^2 - Y^0)(\alpha^1 + \alpha^2)\right\} && (26)
\end{aligned}
$$

In MITC3, the order of the transverse shear strain is reduced to avoid the shear locking. The transverse shear strain ϵ_{rt} and ϵ_{st} can be written using the values at the trying points as:

$$\epsilon_{rt} = \epsilon_{rt}^A + s(\epsilon_{rt}^C - \epsilon_{rt}^A - \epsilon_{st}^C + \epsilon_{st}^B) \quad (27)$$

$$\epsilon_{st} = \epsilon_{st}^B - r(\epsilon_{rt}^C - \epsilon_{rt}^A - \epsilon_{st}^C + \epsilon_{st}^B) \quad (28)$$

Using these coefficients the strain tensor can be written as:

$$\boldsymbol{\epsilon} = \epsilon_{ij}\mathbf{G}^i \otimes \mathbf{G}^j \tag{29}$$

5 Stress and Constitution Tensors

Let us define following two tensors

$$\begin{aligned} \mathbf{I}_2 &= \delta_i^j \mathbf{G}^i \otimes \mathbf{G}_j & (30)\\ &= G^{ij}\mathbf{G}_i \otimes \mathbf{G}_j & (31)\\ \mathbf{I}_4 &= \delta_k^i \delta_l^j \mathbf{G}^k \otimes \mathbf{G}^l \otimes \mathbf{G}_i \otimes \mathbf{G}_j & (32)\\ &= G^{ik}G^{jl}\mathbf{G}_k \otimes \mathbf{G}_l \otimes \mathbf{G}_i \otimes \mathbf{G}_j & (33) \end{aligned}$$

for the second-order tensor $\mathbf{A}$, the inner products for these two tensor are

$$\begin{aligned} tr(\mathbf{A}) &= \mathbf{I}_2 : \mathbf{A} & (34)\\ \mathbf{A} &= \mathbf{I}_4 : \mathbf{A} & (35) \end{aligned}$$

In linear elasticity, the linear stress can be written as:

$$\boldsymbol{\sigma} = \lambda tr(\boldsymbol{\epsilon})\mathbf{I} + 2\mu\boldsymbol{\epsilon} \tag{36}$$

The constitution tensor can be written as:

$$\begin{aligned} \mathbf{C} &= \lambda \mathbf{I}_2 \otimes \mathbf{I}_2 + 2\mu \mathbf{I}_4 & (37)\\ &= (\lambda G^{ij}G^{kl} + \mu G^{ik}G^{jl} + \mu G^{il}G^{jk})\mathbf{G}_k \otimes \mathbf{G}_l \otimes \mathbf{G}_i \otimes \mathbf{G}_j & (38)\\ &= C^{ijkl} & (39) \end{aligned}$$

The stress can be written as

$$\boldsymbol{\sigma} = \sigma^{ij}\mathbf{G}_i \otimes \mathbf{G}_j, \;\; where \;\; \sigma^{ij} = C^{ijkl}\epsilon_{kl} \tag{40}$$

the elastic potential energy density can be written as:

$$\boldsymbol{\sigma} : \boldsymbol{\epsilon} = \epsilon_{ij}C^{ijkl}\epsilon_{kl} \tag{41}$$

6 Vector Notations of Tensors

we define a strain vector $\mathbf{e}$ and the stress vector $\mathbf{s}$ as

$$\mathbf{e} = \{\epsilon_{rr}, \epsilon_{ss}, \epsilon_{rs}, \epsilon_{tr}, \epsilon_{ts}\} \tag{42}$$

$$\mathbf{s} = \{\sigma_{rr}, \sigma_{ss}, 2\sigma_{rs}, 2\sigma_{tr}, 2\sigma_{ts}\} \tag{43}$$

with this definition, the energy density can be written with the dot product $\mathbf{e}\cdot\mathbf{s}$. The constitute matrix $\mathbf{C}$, where $\mathbf{s}=\mathbf{Ce}$ is computed as

$$\mathbf{C}=\begin{bmatrix} C^{rrrr} & C^{rrss} & 2C^{rrrs} & 2C^{rrrt} & 2C^{rrrs} \\ C^{ssrr} & C^{ssss} & 2C^{ssrs} & 2C^{ssrt} & 2C^{ssrs} \\ 2C^{rsrr} & 2C^{rsss} & 4C^{rsrs} & 4C^{rsrt} & 4C^{rsrs} \\ 2C^{rtrr} & 2C^{rtss} & 4C^{rtrs} & 4C^{rtrt} & 4C^{rtrs} \\ 2C^{strr} & 2C^{stss} & 4C^{strs} & 4C^{strt} & 4C^{strs} \end{bmatrix} \tag{44}$$

7 Mass Matrix

We need mass matrix for a dynamic analysis. Let us consider the lumped mass matrix for

Using the deformation (**??**)

$$\frac{\rho}{2}\int_{-a/2}^{+a/2}(\mathbf{u}\cdot\mathbf{u})da = \frac{\rho}{2}\int_{-a/2}^{+a/2}(a\beta)^2+(a\alpha)^2+(u_z)^2da \tag{45}$$

$$= \frac{\rho}{2}\left(\frac{a^3}{12}\beta^2+\frac{a^3}{12}\alpha^2+au_z^2\right) \tag{46}$$

Hence the mass matrix when the degree of freedom at the node $\{u_z,\alpha,\beta\}$ becomes

$$\rho A_i\begin{bmatrix} a & 0 & 0 \\ 0 & a^3/12 & 0 \\ 0 & 0 & a^3/12 \end{bmatrix}, \tag{47}$$

where A_i is the area of the mesh belong to the node i.