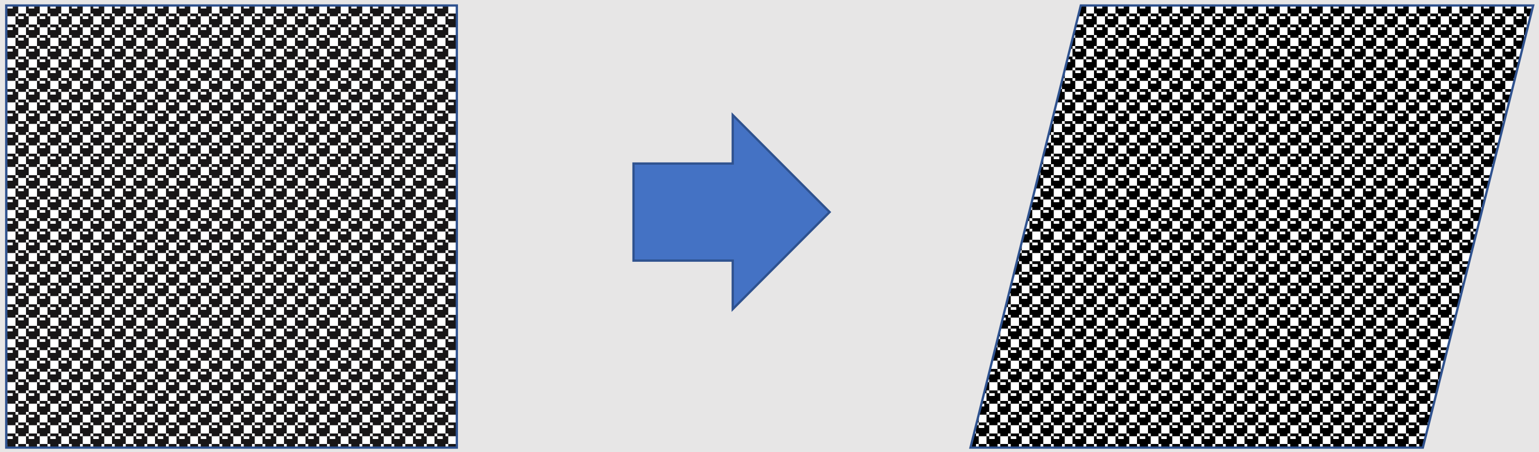
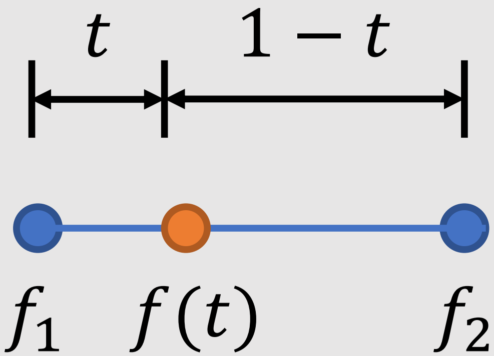
Grid & Mesh Interpolation

Continuum Approximation

- Drastically reducing degrees of freedom (DoFs)



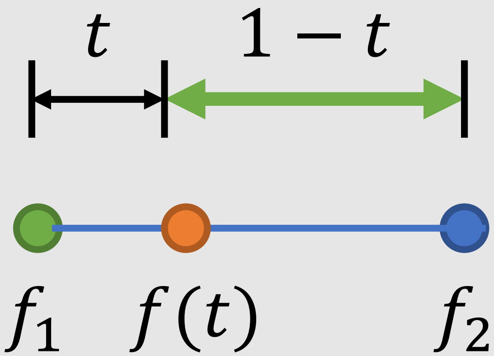
Linear Interpolation (1D)



$$f(0) = f_1$$

$$f(1) = f_2$$

Linear Interpolation (1D)



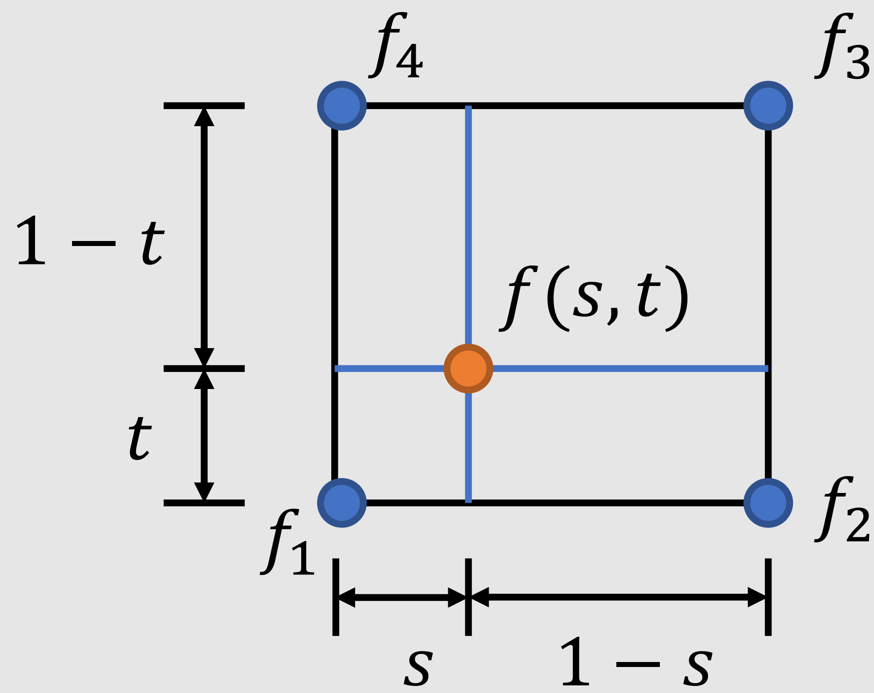
$$f(0) = f_1$$

$$f(1) = f_2$$

Weights sums to 1

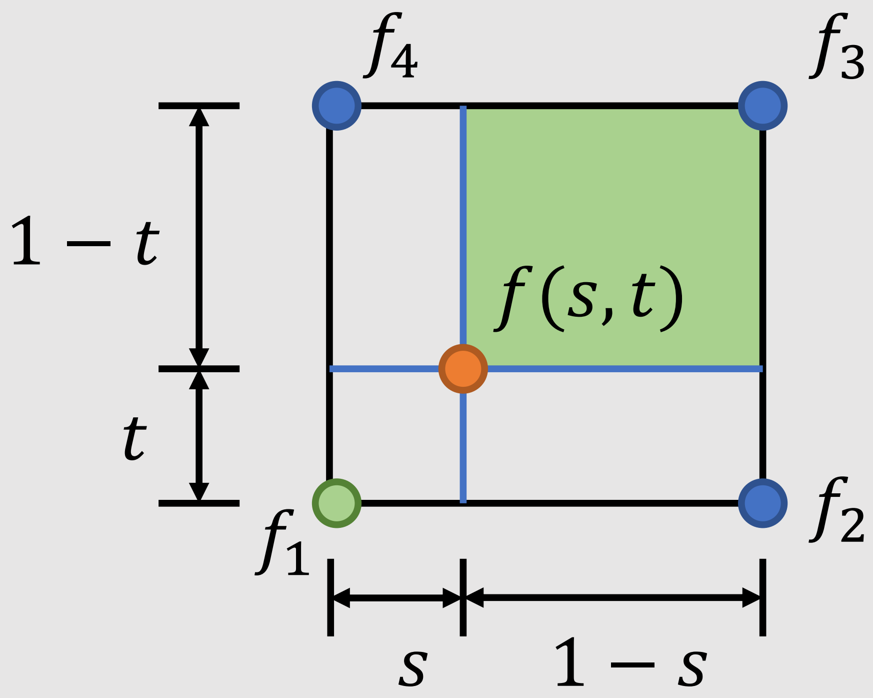
$$f(t) = \boxed{(1-t)} f_1 + t f_2$$

Bilinear Interpolation (2D)



$$f(0,0) = f_1$$
$$f(1,0) = f_2$$
$$f(1,1) = f_3$$
$$f(0,1) = f_4$$

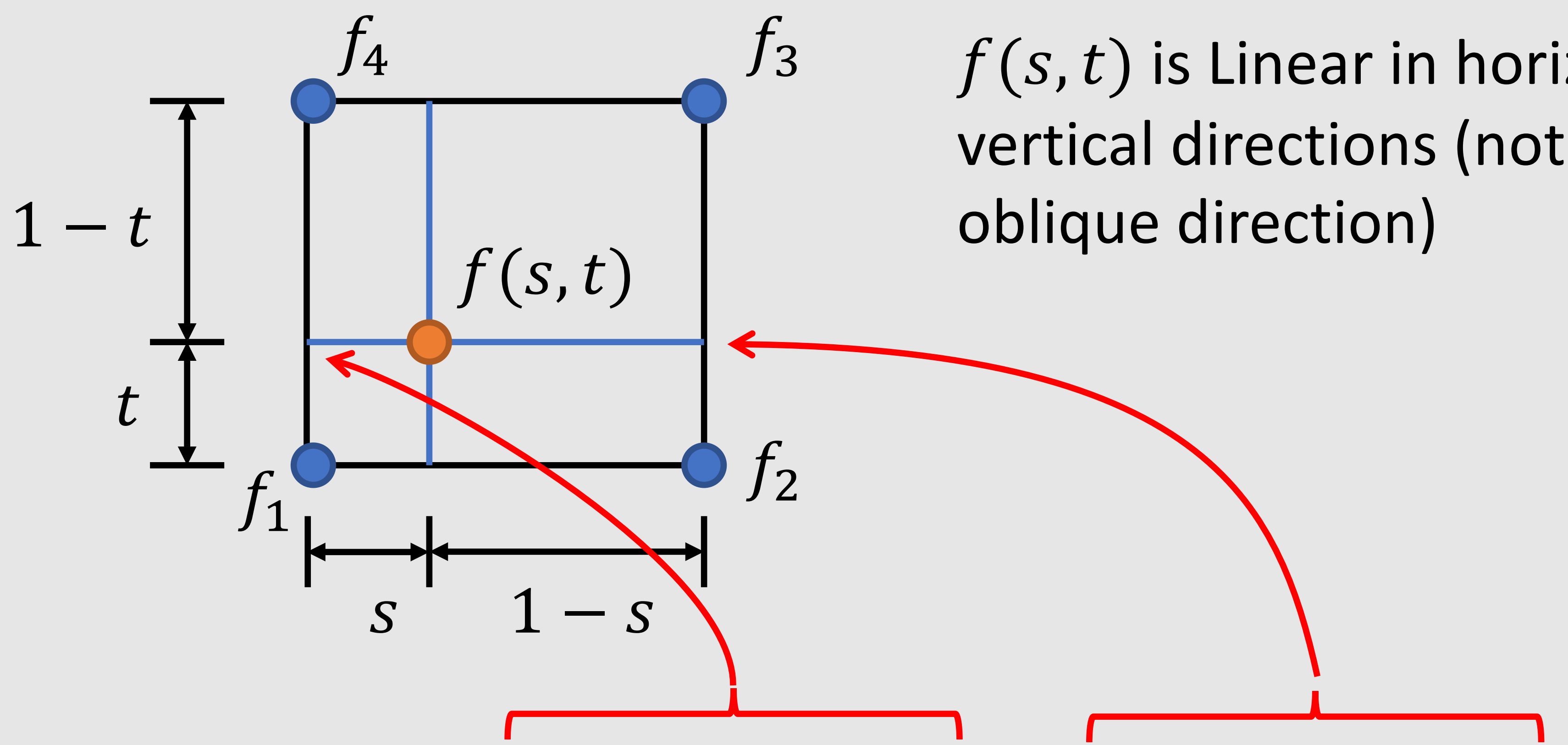
Bilinear Interpolation (2D)



Weights sums to 1

$$f(s,t) = \boxed{(1-s)(1-t)f_1} + s(1-t)f_2 + stf_3 + (1-s)tf_4$$

Bilinear Interpolation (2D)

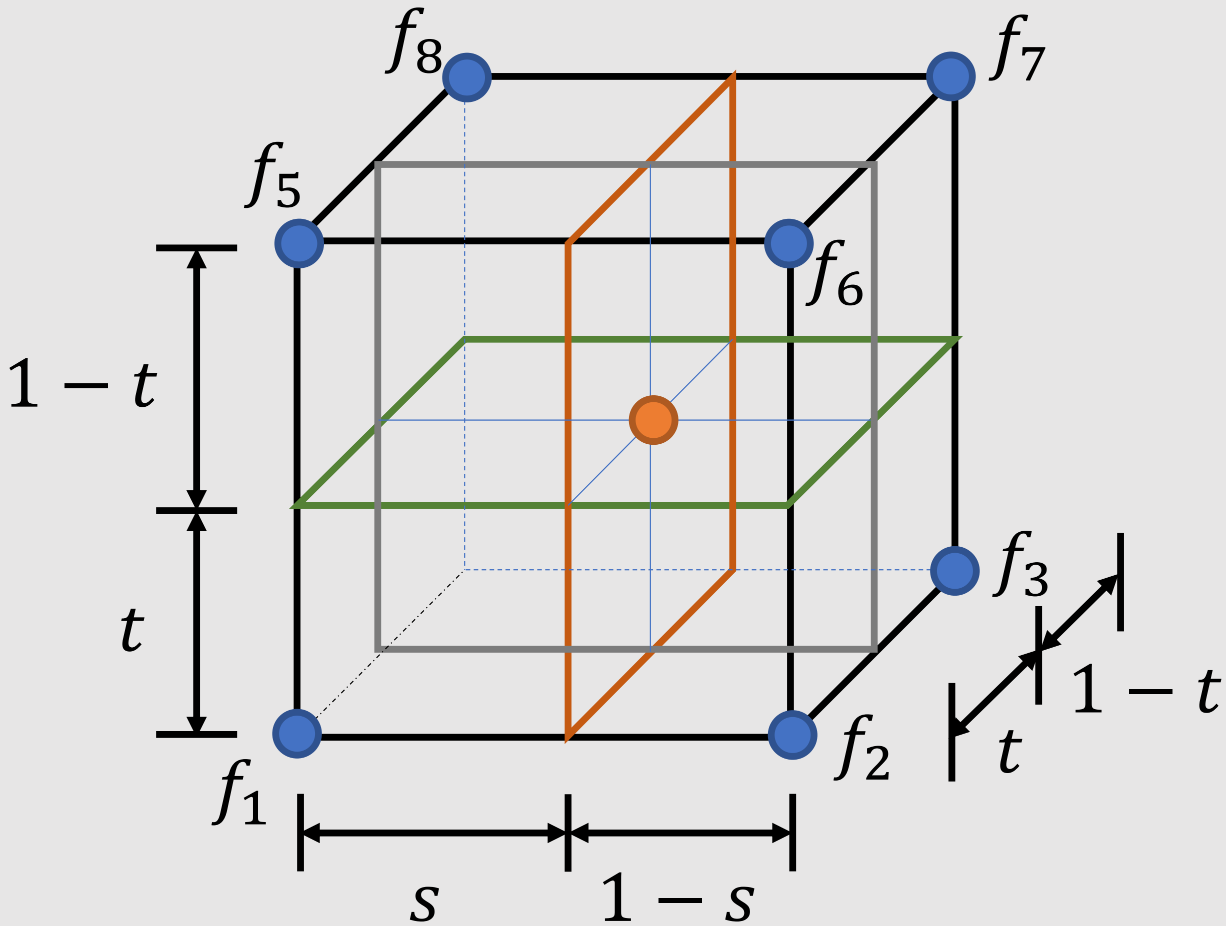


$f(s,t)$ is Linear in horizontal & vertical directions (not in an oblique direction)

$$f(s,t) = (1-s)[(1-t)f_1 + tf_4] + s[(1-t)f_2 + tf_3]$$

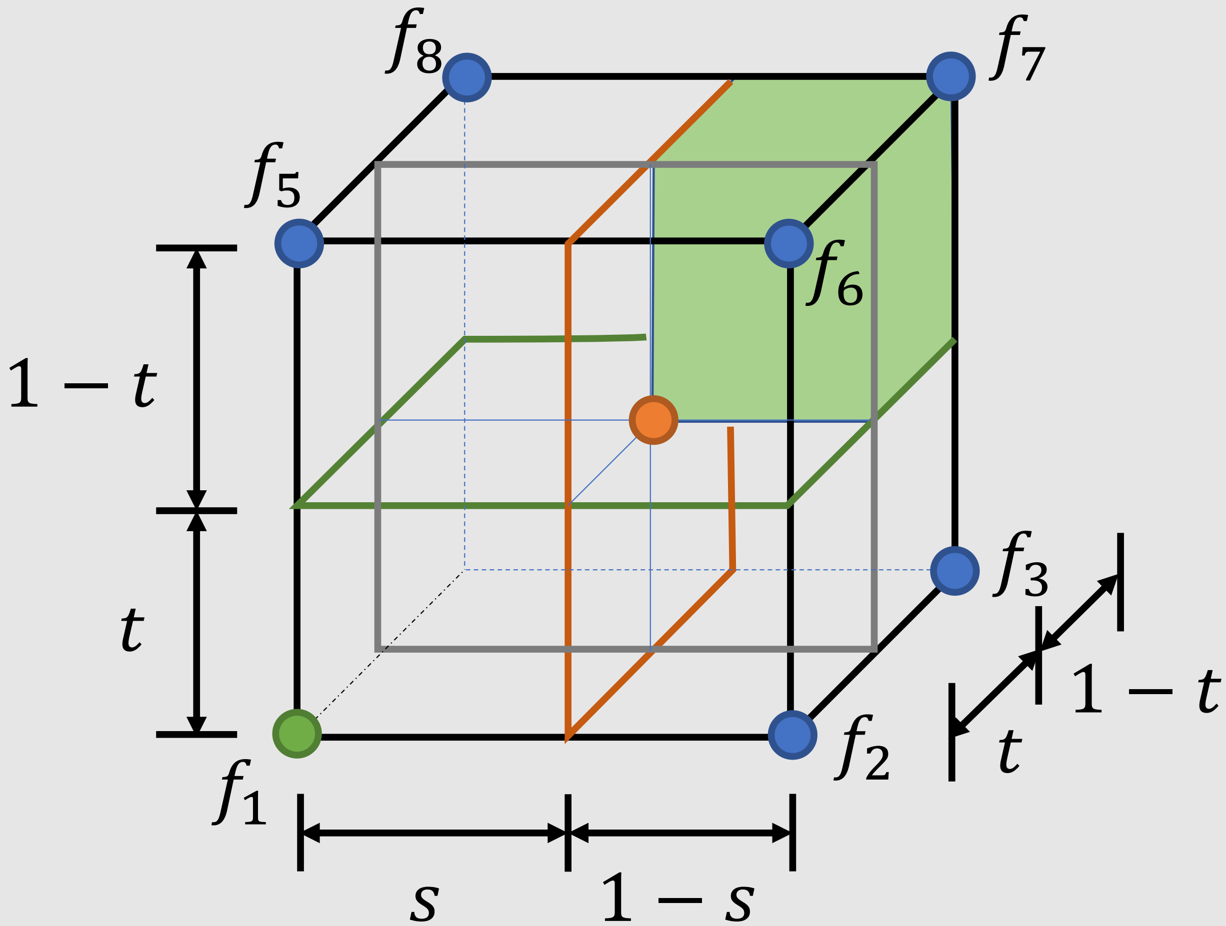
Trilinear Interpolation (3D)

$$
\begin{aligned}
f(s,t,u) = \\
&(1-s)(1-t)(1-u)f_1 + \\
&s(1-t)(1-u)f_2 + \\
&st(1-u)f_3 + \\
&(1-s)t(1-u)f_4 + \\
&(1-s)(1-t)uf_5 + \\
&s(1-t)uf_6 + \\
&stuf_7 + \\
&(1-s)tuf_8
\end{aligned}
$$



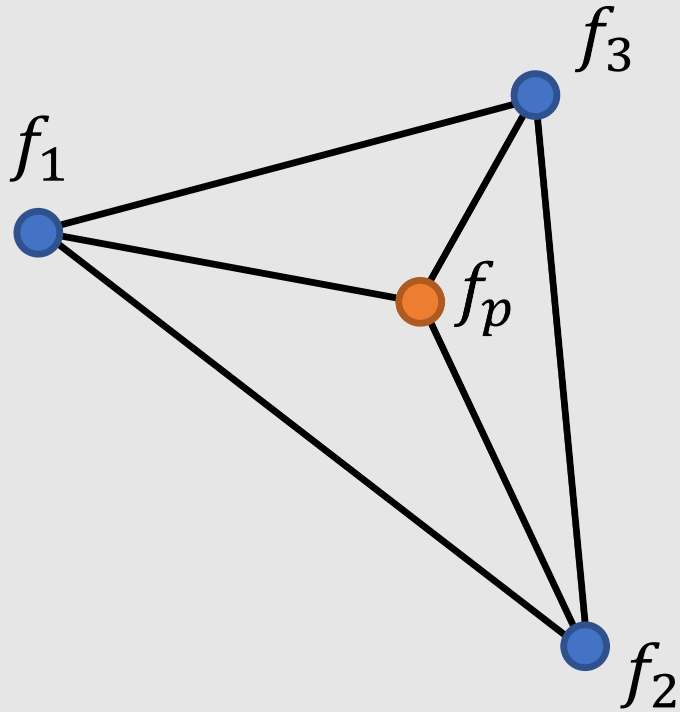
Trilinear Interpolation (3D)

$$f(s,t,u) =$$
$$(1-s)(1-t)(1-u)f_1 +$$
$$s(1-t)(1-u)f_2 +$$
$$st(1-u)f_3 +$$
$$(1-s)t(1-u)f_4 +$$
$$(1-s)(1-t)uf_5 +$$
$$s(1-t)uf_6 +$$
$$stuf_7 +$$
$$(1-s)tuf_8$$



Barycentric Coordinates for a Triangle

$$f_p = f(L_1, L_2, L_3) = L_1 f_1 + L_2 f_2 + L_3 f_3$$



$$L_1 = \frac{Area(p, 2,3)}{Area(1,2,3)}$$

$$L_2 = \frac{Area(1, p, 3)}{Area(1,2,3)}$$

$$L_3 = \frac{Area(1,2, p)}{Area(1,2,3)}$$

Barycentric Coordinates for a Tetrahedron

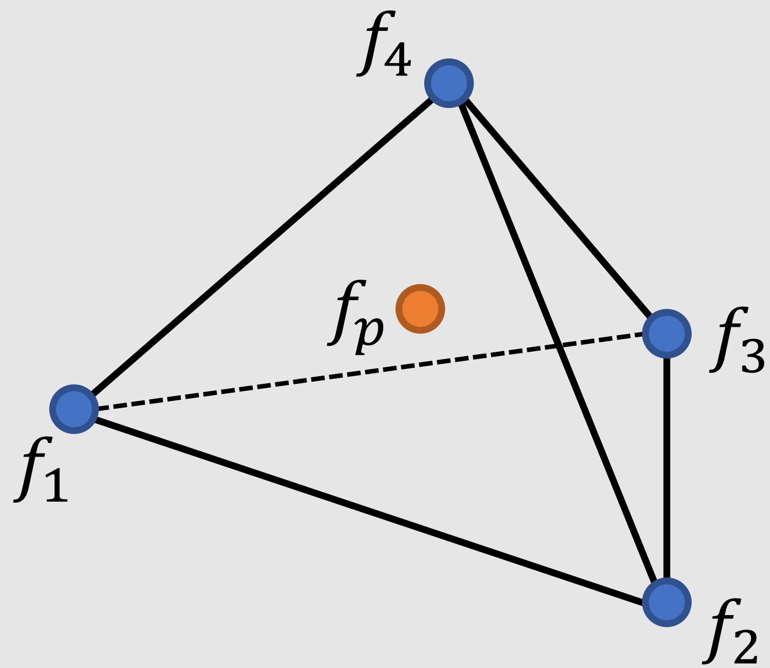
$$f_p = f(L_1, L_2, L_3, L_4)$$
$$= L_1 f_1 + L_2 f_2 + L_3 f_3 + L_4 f_4$$

$$L_1 = \frac{Volume(p,2,3,4)}{Volume(1,2,3,4)}$$

$$L_2 = \frac{Volume(1,p,3,4)}{Volume(1,2,3,4)}$$

$$L_3 = \frac{Volume(1,2,p,4)}{Volume(1,2,3,4)}$$

$$L_4 = \frac{Volume(1,2,3,p)}{Volume(1,2,3,4)}$$



Barycentric Coordinates for a Tetrahedron

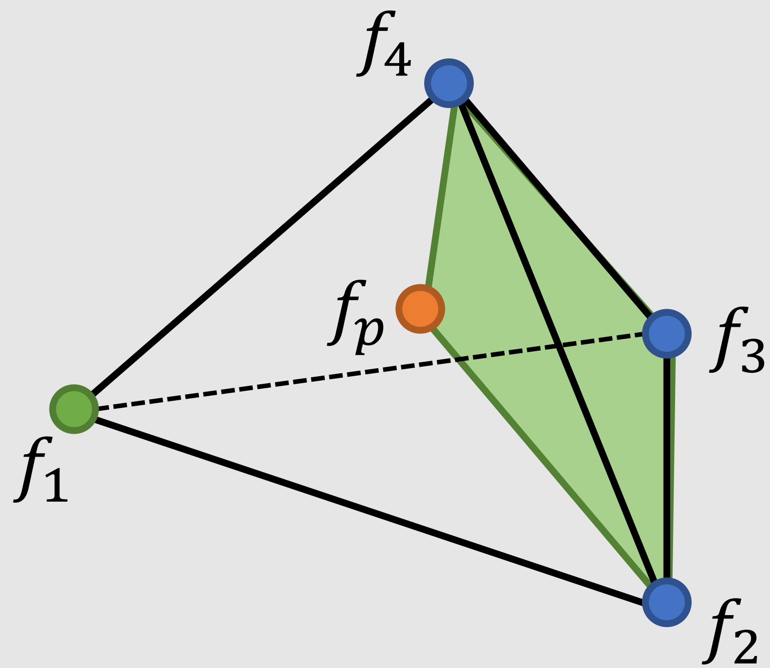
$$f_p = f(L_1, L_2, L_3, L_4)$$
$$= L_1 f_1 + L_2 f_2 + L_3 f_3 + L_4 f_4$$

$$L_1 = \frac{Volume(p,2,3,4)}{Volume(1,2,3,4)}$$

$$L_2 = \frac{Volume(1,p,3,4)}{Volume(1,2,3,4)}$$

$$L_3 = \frac{Volume(1,2,p,4)}{Volume(1,2,3,4)}$$

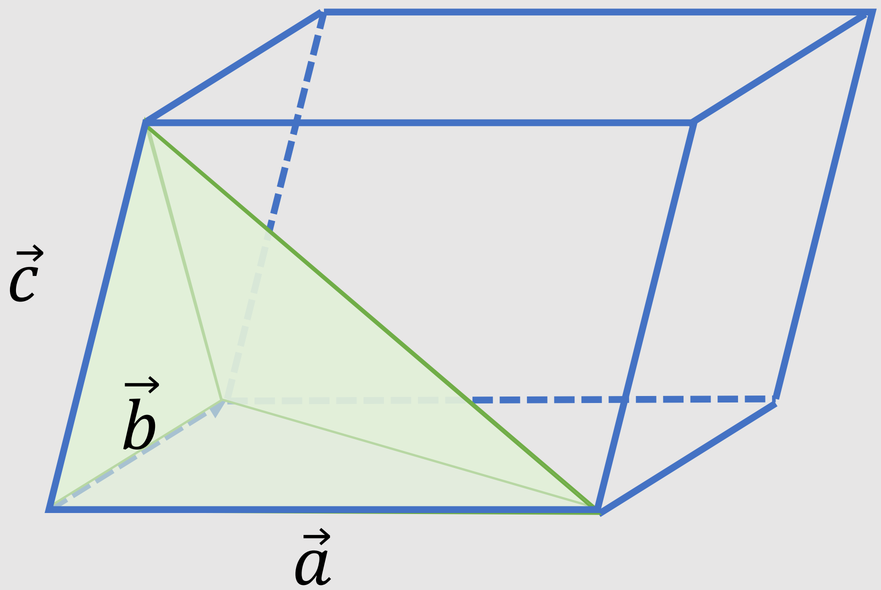
$$L_4 = \frac{Volume(1,2,3,p)}{Volume(1,2,3,4)}$$



Differentiation of Interpolated Values

Volume of Tetrahedron from Parallelepiped

- Volume of parallelepiped: $\mathrm{V} = \vec{a} \cdot (\vec{b} \times \vec{c})$
- Volume of tetrahedron: $\mathrm{V} = 1/6\, \vec{a} \cdot (\vec{b} \times \vec{c})$

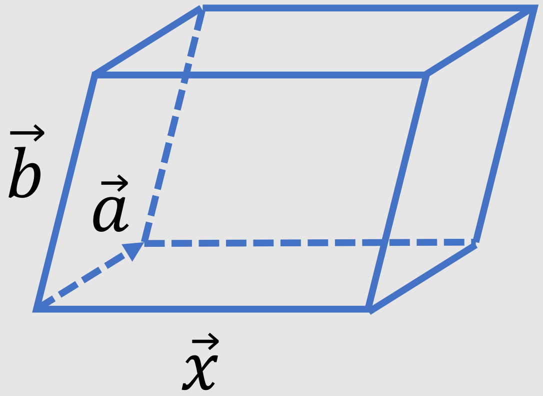


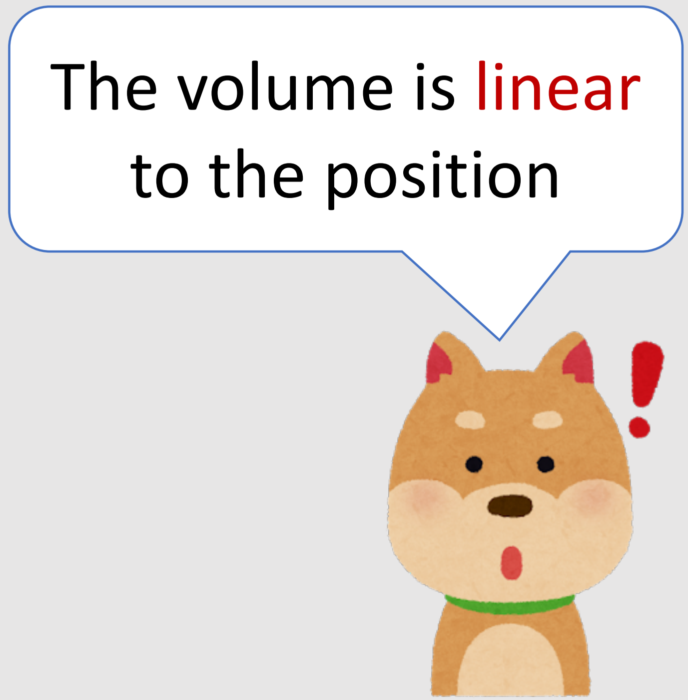
Differential of Scalar Triple Product

$$W(\vec{x}) = \vec{b} \cdot (\vec{a} \times \vec{x})$$

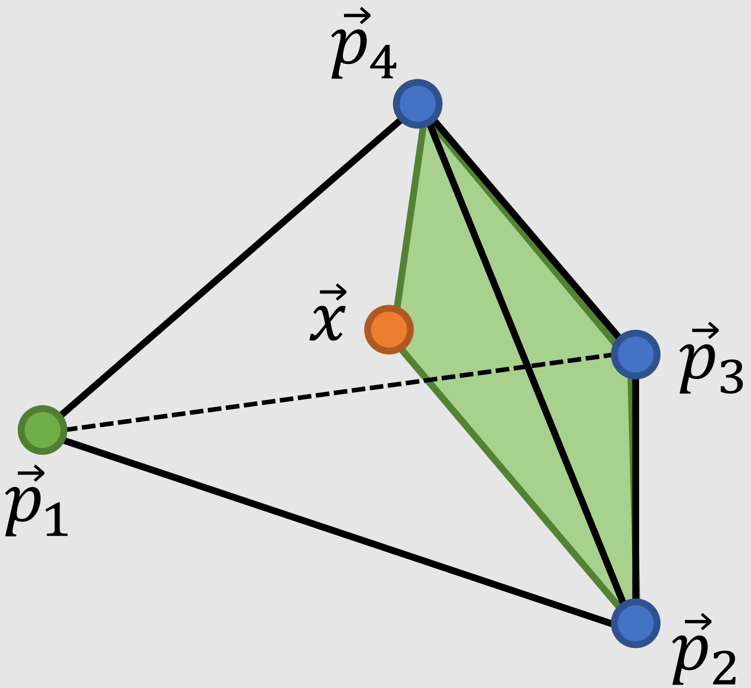
$$= \vec{x} \cdot (\vec{b} \times \vec{a})$$

$$\frac{\partial W}{\partial \vec{x}} = \vec{b} \times \vec{a}$$





Volume of Tetrahedron



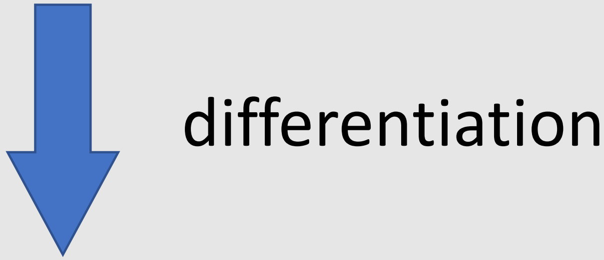
$$L_1 = \frac{Volume(\vec{x}, \vec{p}_2, \vec{p}_3, \vec{p}_4)}{Volume(\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4)}$$

$$Volume(\vec{x}, \vec{p}_2, \vec{p}_3, \vec{p}_4)$$
$$= 1/6\ (\vec{x} - \vec{p}_2)\ \cdot \{(\vec{p}_4 - \vec{p}_2) \times (\vec{p}_3 - \vec{p}_2)\}$$

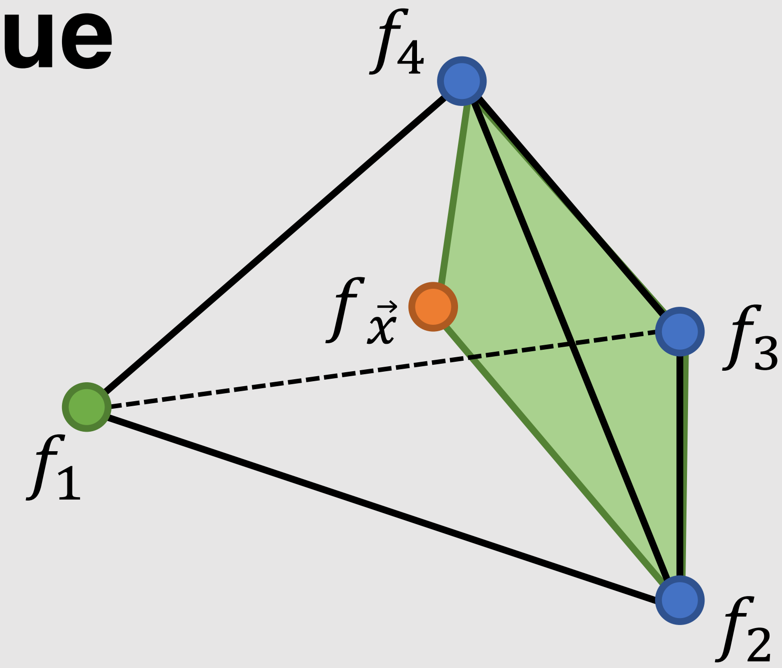
$$\frac{\partial L_1}{\partial \vec{x}} = \frac{1/6\,(\vec{p}_4 - \vec{p}_2) \times (\vec{p}_3 - \vec{p}_2)}{Volume(\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4)}$$

Gradient of Interpolated Value

$$f_{\vec{x}} = f(L_1, L_2, L_3, L_4)$$
$$= L_1 f_1 + L_2 f_2 + L_3 f_3 + L_4 f_4$$



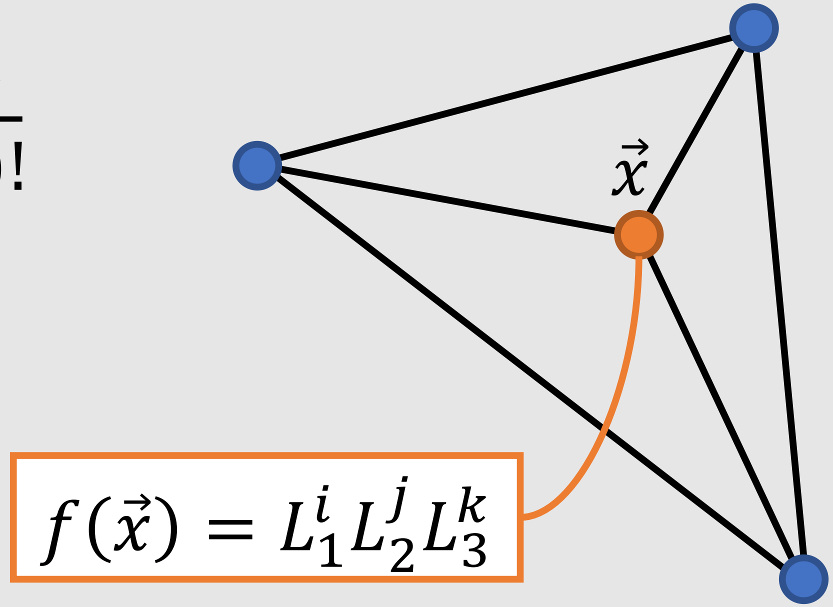
$$\frac{\partial f_{\vec{x}}}{\partial \vec{x}} = \frac{\partial L_1}{\partial \vec{x}} f_1 + \frac{\partial L_2}{\partial \vec{x}} f_2 + \frac{\partial L_3}{\partial \vec{x}} f_3 + \frac{\partial L_4}{\partial \vec{x}} f_4$$



Integration of Polynomial over Simplex

Integration Rule for Triangle

$$\int_{\vec{x}\in Tri} L_1^i L_2^j L_3^k d\vec{x} = \frac{2!\, i!\, j!\, k!\, Area}{(2+i+j+k)!}$$



Baldoni, Velleda, Nicole Berline, Jesus De Loera, Matthias Köppe, and Michèle Vergne. "How to integrate a polynomial over a simplex." *Mathematics of Computation* 80, no. 273 (2011): 297–325. https://arxiv.org/abs/0809.2083

Example of Integration over Triangle

$$\int_{\vec{x}\in Tri} L_1 d\vec{x} = \int_{\vec{x}\in Tri} L_1^1 L_2^0 L_3^0 d\vec{x} = \frac{2!\,1!\,0!\,0!\,Area}{(2+1+0+0)!} = \frac{1}{3}Area$$

$$\int_{\vec{x}\in Tri} L_1^2 d\vec{x} = \int_{\vec{x}\in Tri} L_1^2 L_2^0 L_3^0 d\vec{x} = \frac{2!\,2!\,0!\,0!\,Area}{(2+2+0+0)!} = \frac{1}{6}Area$$

$$\int_{\vec{x}\in Tri} L_1 L_2 d\vec{x} = \int_{\vec{x}\in Tri} L_1^1 L_2^1 L_3^0 d\vec{x} = \frac{2!\,1!\,1!\,0!\,Area}{(2+1+1+0)!} = \frac{1}{12}Area$$

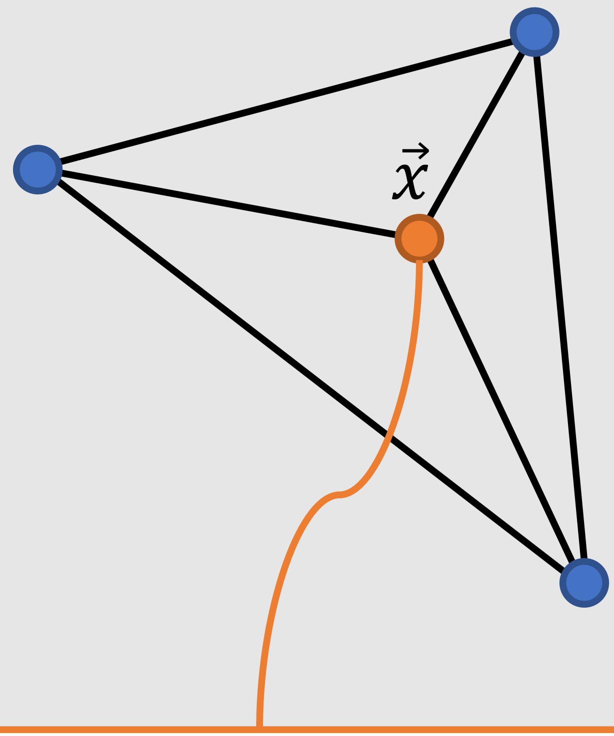
$$\int_{\vec{x}\in Tri} L_a L_b d\vec{x} = \frac{1}{12}Area(1+\delta_{ab})$$

Center of The Gravity of a Triangle in 3D

$$\vec{p}_{cg} = \frac{1}{Area}\int_{\vec{x}\in \mathrm{Tri}} \vec{x}\, d\vec{x}$$

$$= \frac{1}{Area}\int_{\vec{x}\in \mathrm{Tri}} L_1\vec{p}_1 + L_2\vec{p}_2 + L_3\vec{p}_3\, d\vec{x}$$

$$= \frac{1}{3}(\vec{p}_1 + \vec{p}_2 + \vec{p}_3)$$



$$\vec{x} = L_1\vec{p}_1 + L_2\vec{p}_2 + L_3\vec{p}_3$$

Inertia Tensor of a Triangle in 3D

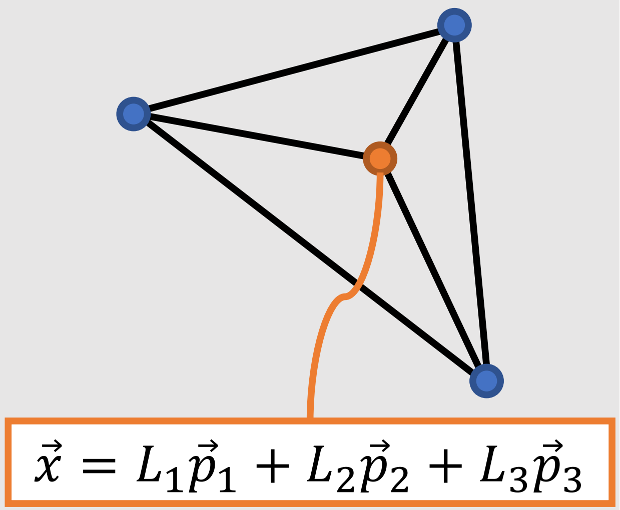
$$I_{in} = \int_{\vec{x}\in\mathrm{Tri}} -\mathrm{Skew}(\vec{x})\mathrm{Skew}(\vec{x})d\vec{x}$$

$$= \int_{\vec{x}\in\mathrm{Tri}} -\mathrm{Skew}(L_1\vec{p}_1 + L_2\vec{p}_2 + L_3\vec{p}_3)\mathrm{Skew}(L_1\vec{p}_1 + L_2\vec{p}_2 + L_3\vec{p}_3)d\vec{x}$$

$$= \int_{\vec{x}\in\mathrm{Tri}} \sum_{1\le a\le 3} \sum_{1\le b\le 3} -\mathrm{Skew}(L_a\vec{p}_b)\mathrm{Skew}(L_a\vec{p}_b)d\vec{x}$$

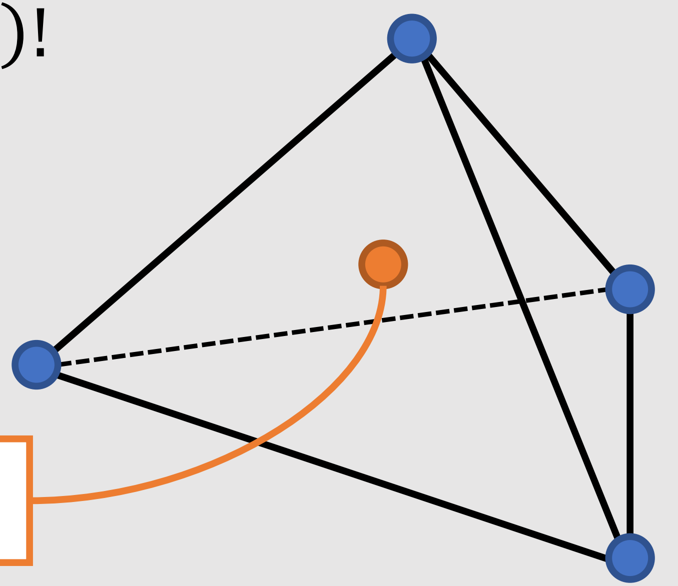
$$= \sum_{1\le a\le 3} \sum_{1\le b\le 3} -\mathrm{Skew}(\vec{p}_a)\mathrm{Skew}(\vec{p}_b) \int_{\vec{x}\in\mathrm{Tri}} L_a L_b d\vec{x}$$

$$= \sum_{1\le a\le 3} \sum_{1\le b\le 3} (\vec{p}_a^T\vec{p}_b I - \vec{p}_a \otimes \vec{p}_b)\frac{Area}{12}(1 + \delta_{ab})$$



Integration Rule for Tetrahedron

$$\int_{\vec{x}\in \mathrm{Tet}} L_1^i L_2^j L_3^k L_4^l d\vec{x} = \frac{3!\, i!\, j!\, k!\, l!\, Volume}{(3+i+j+k+l)!}$$



$$\vec{f}(\vec{x}) = L_1^i L_2^j L_3^k L_4^l$$

Baldoni, Velleda, Nicole Berline, Jesus De Loera, Matthias Köppe, and Michèle Vergne. "How to integrate a polynomial over a simplex." *Mathematics of Computation* 80, no. 273 (2011): 297–325. https://arxiv.org/abs/0809.2083

Example of Integration over Tetrahedra

$$\int_{\vec{x}\in Tet} L_1 d\vec{x} = \int_{\vec{x}\in Tri} L_1^1 L_2^0 L_3^0 L_4^0 d\vec{x} = \frac{3!\,1!\,0!\,0!\,0!\,Volume}{(3+1+0+0+0)!} = \frac{1}{4} Volume$$

$$\int_{\vec{x}\in Tet} L_1^2 d\vec{x} = \int_{\vec{x}\in Tri} L_1^2 L_2^0 L_3^0 L_4^0 d\vec{x} = \frac{3!\,2!\,0!\,0!\,0!\,Volume}{(3+2+0+0+0)!} = \frac{1}{10} Volume$$

$$\int_{\vec{x}\in Tet} L_1 L_2 d\vec{x} = \int_{\vec{x}\in Tri} L_1^1 L_2^1 L_3^0 L_4^0 d\vec{x} = \frac{2!\,1!\,1!\,0!\,0!\,Volume}{(3+1+1+0+0)!} = \frac{1}{20} Volume$$

$$\int_{\vec{x}\in Tet} L_a L_b d\vec{x} = \frac{1}{20} Volume(1+\delta_{ab})$$

Inertia Tensor of a Tetrahedron

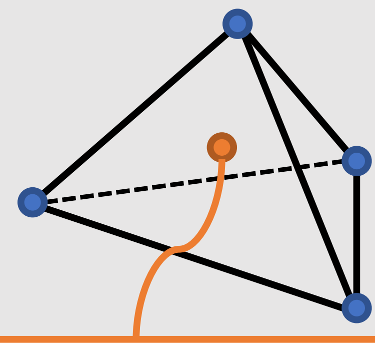
$$I_{in} = \int_{\vec{x}\in \mathrm{Tet}} -\mathrm{Skew}(\vec{x})\mathrm{Skew}(\vec{x})d\vec{x}$$

$$= \int_{\vec{x}\in \mathrm{Tet}} -\mathrm{Skew}(L_1\vec{p}_1 + L_2\vec{p}_2 + L_3\vec{p}_3 + L_4\vec{p}_4)\mathrm{Skew}(L_1\vec{p}_1 + L_2\vec{p}_2 + L_3\vec{p}_3 + L_4\vec{p}_4)d\vec{x}$$

$$= \int_{\vec{x}\in \mathrm{Tet}} \sum_{1\le a\le 4} \sum_{1\le b\le 4} -\mathrm{Skew}(L_a\vec{p}_b)\mathrm{Skew}(L_a\vec{p}_b)d\vec{x}$$

$$= \sum_{1\le a\le 4} \sum_{1\le b\le 4} -\mathrm{Skew}(\vec{p}_a)\mathrm{Skew}(\vec{p}_b) \int_{\vec{x}\in \mathrm{Tet}} L_a L_b d\vec{x}$$

$$= \sum_{1\le a\le 4} \sum_{1\le b\le 4} (\vec{p}_a^T\vec{p}_b I - \vec{p}_a \otimes \vec{p}_b)\frac{Volume}{20}(1+\delta_{ab})$$

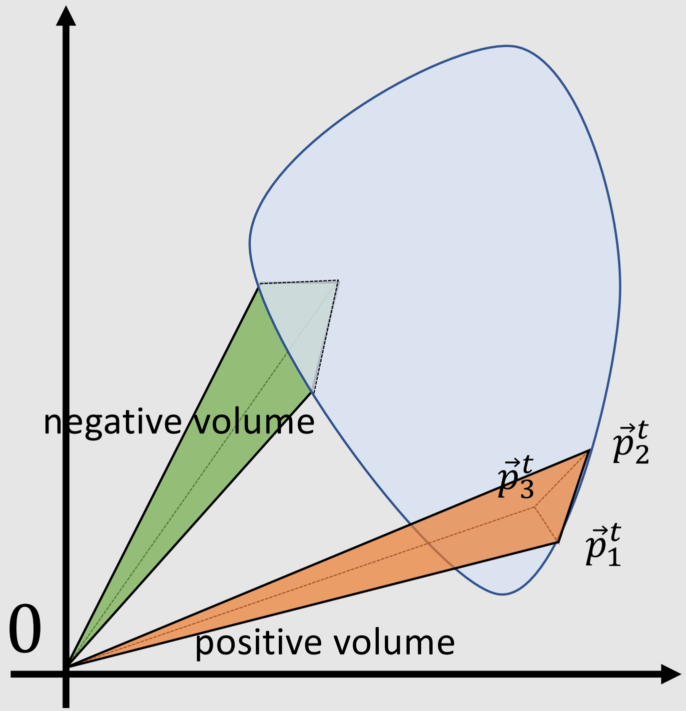


$\vec{x} = L_1\vec{p}_1 + L_2\vec{p}_2 + L_3\vec{p}_3 + L_4\vec{p}_4$

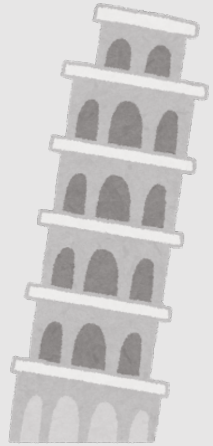
Inertia Tensor of a Solid 3D Triangle Mesh

- Summing over inertia tensors of tetrahedra connecting the origin of the coordinate and the three corner points of the triangle

$$I_{inertia} = \sum_{t \in Tri} I_{inertia}(0, \vec{p}_1^t, \vec{p}_2^t, \vec{p}_3^t)$$

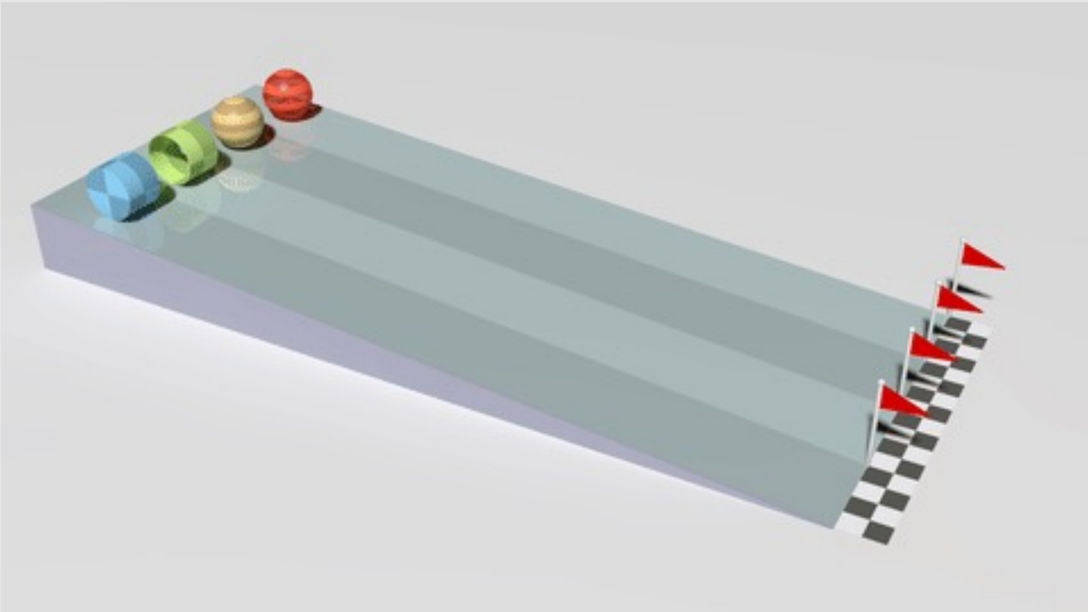


Inertia Tensor = How Hard to Rotate



Same radius, same mass,
different speed!?

Galileo Galilei



Credit: Lucas Vieira @ Wikipedia

$$\mathcal{K} = \frac{1}{2}M\left\|\vec{v}_{cg}\right\|^2 + \frac{1}{2}\vec{\Omega}^T I_{in}\vec{\Omega}$$