Lagrangian Mechanics

Pros & Cons

Newton's mechanics

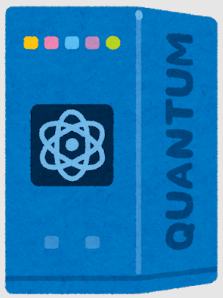
- ☺ Easy to understand

- ☹ Only takes Cartesian coordinate values
- ☹ System of linear equations

Lagrangian Mechanics

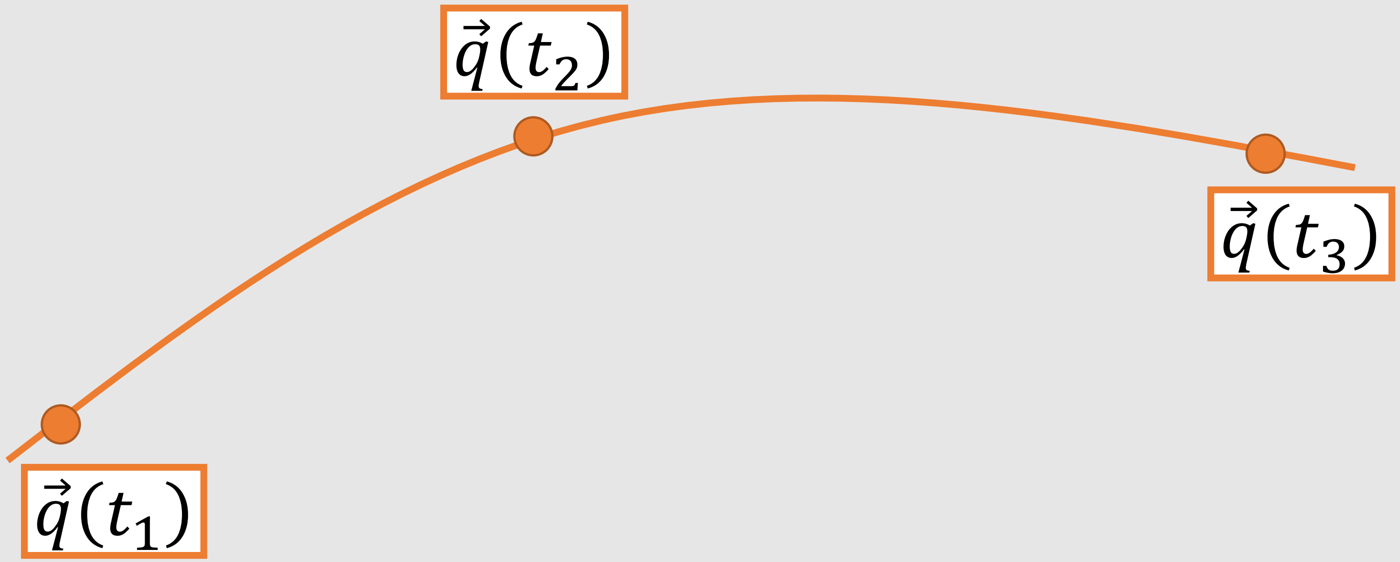
- ☹ Hard to understand

- ☺ Arbitrary variables (e.g., angles)
- ☺ Optimization of scalar value
- ☺ Leading to modern physics
- ☺ Simple & beautiful



Trajectory of State

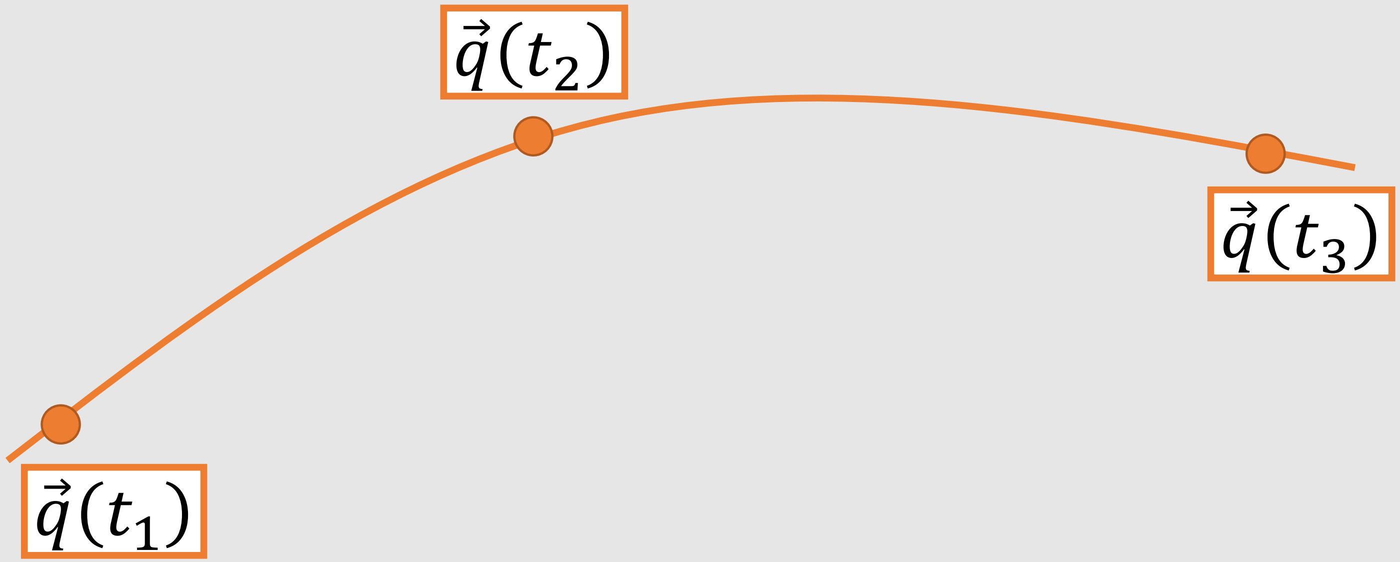
- State $\vec{q}$ is the parameters to represent physics phenomena written in any coordinate systems



Lagrangian: Kinetic Enrgy – Potential Enrgy

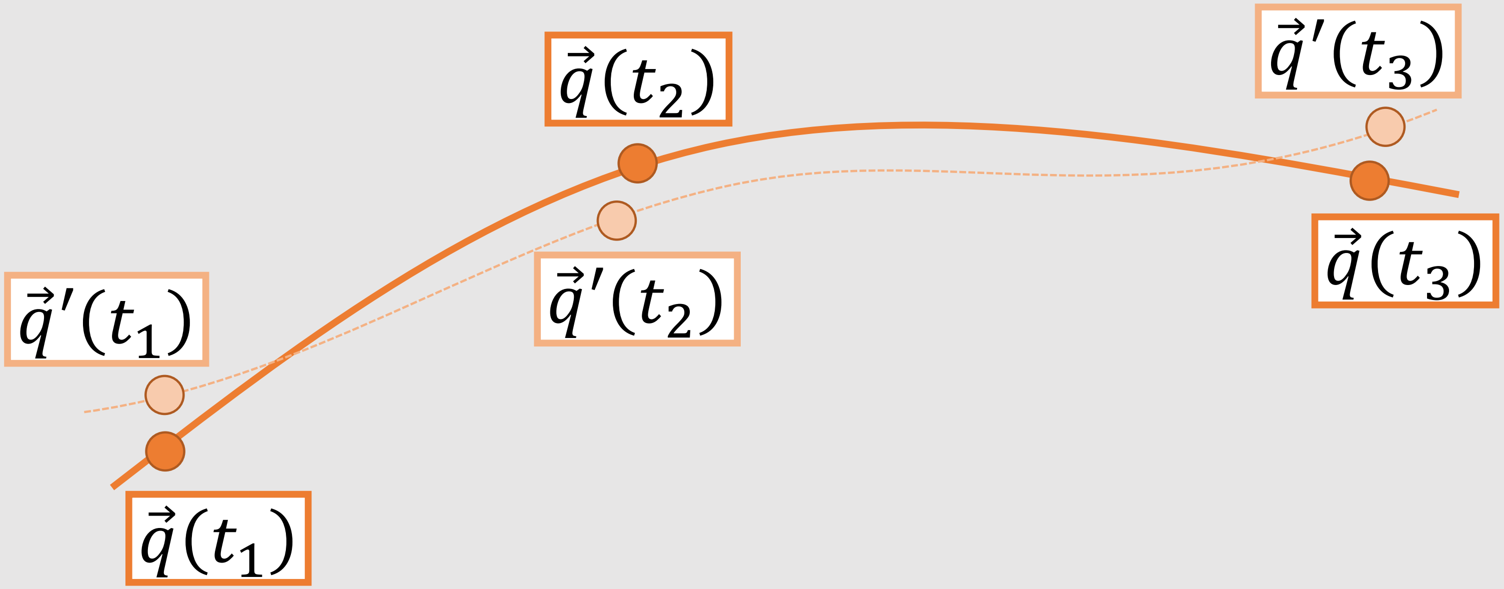
kinetic energy   potential energy

$$\mathcal{L}(\vec{q}, \dot{\vec{q}}) = \mathcal{K} - \mathcal{P}$$



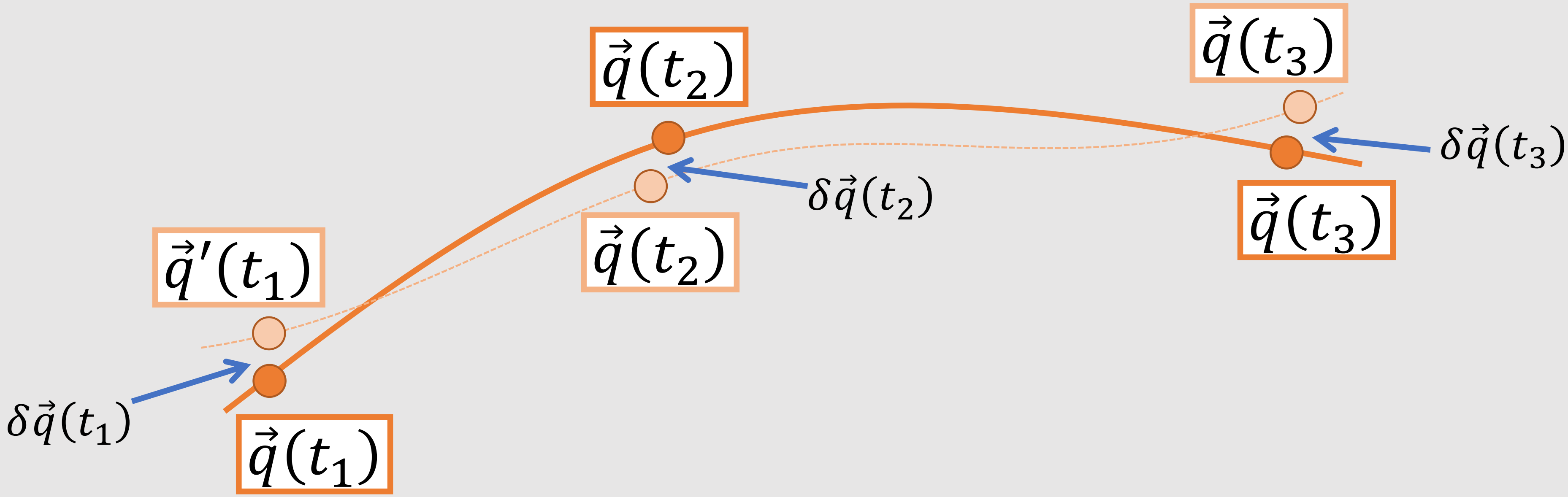
Perturbation of State

Lagrangian for perturbed state: $\mathcal{L}'(\vec{q}', \dot{\vec{q}}')$



Perturbation of Lagrangian

$$\delta\mathcal{L}(\vec{q},\dot{\vec{q}},\delta\vec{q},\delta\dot{\vec{q}}) = \mathcal{L}'(\vec{q}',\dot{\vec{q}}') - \mathcal{L}(\vec{q},\dot{\vec{q}})$$



Euler-Lagrange Equation

- If $\vec{q}(t)$ is the solution, for arbitrary perturbation $\delta\vec{q}(t)$ it holds:

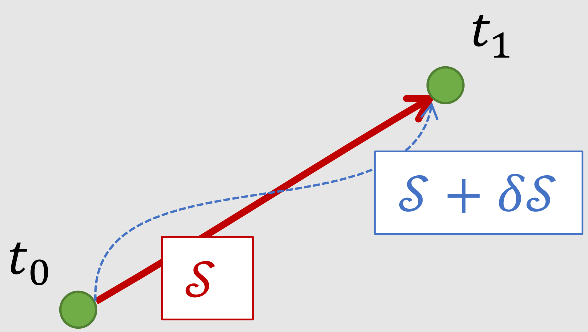
$$\frac{d}{dt}\left(\frac{\partial \delta \mathcal{L}}{\partial \delta \dot{\vec{q}}}\right) - \frac{\partial \delta \mathcal{L}}{\partial \delta \vec{q}} = 0$$

What Euler-Lagrange Equation Means?

- Action $\mathcal{S}$: time integration of Lagrangian $\mathcal{L}$

$$\mathcal{S}(q, \dot{q}) = \int_{t_0}^{t_1} \mathcal{L}(q, \dot{q})\, dt$$

- Hamilton's principle: perturbation of action is zero $\delta\mathcal{S} = 0$



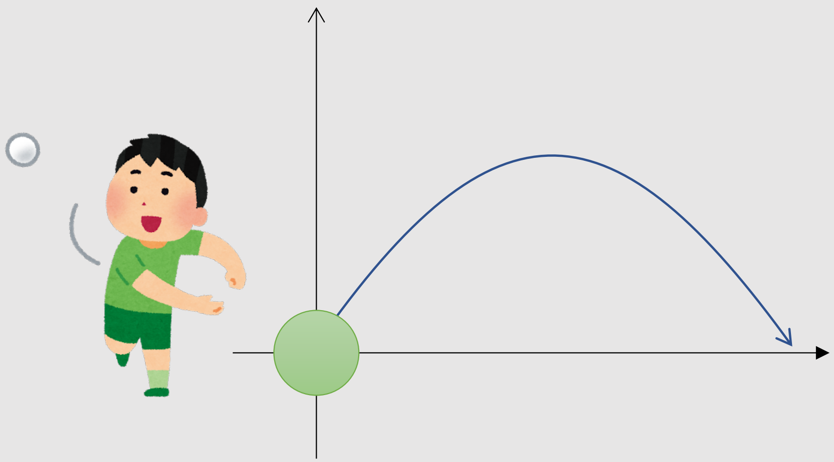
Hamilton’s Principle: Minimal Action

Minimizing time integration of Lagrangian $\mathcal{L}(q, \dot{q}) = \mathcal{K} - \mathcal{W}$

→ Minimize $\mathcal{K}$ and maximize $\mathcal{W}$

smooth motion (Minimize $\mathcal{K}$)

movement is slow when potential energy is large (maximize $\mathcal{W}$)



why making potential $\mathcal{W}$ large instead of small?

Symmetry in Lagrangian = Conservation Law

- Noether's Theorem
- Symmetry = invariance under operation

Emmy Noether

translational symmetry ⟺ conservation of linear momentum

translational symmetry ⟺ conservation of angular momentum

temporal symmetry ⟺ conservation of energy

Example of Lagrange's Equation of Motion

Falling ball

$$\mathcal{L}(q, \dot{q}) = \mathcal{K} - \mathcal{W}$$

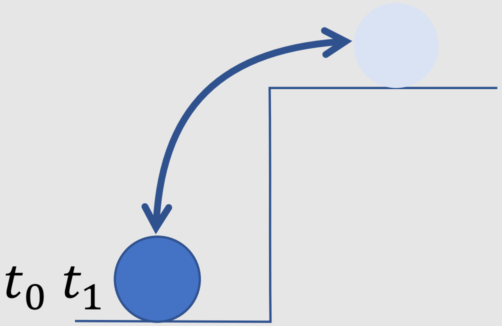
$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = 0$$

(Write Equations Here)

Hamilton's Principle is *Unintuitive*

Making Lagrangian $\mathcal{L}(q, \dot{q}) = \mathcal{K} - \mathcal{W}$ small?

can a ball climb up a cliff ?



can a ball float in air ?