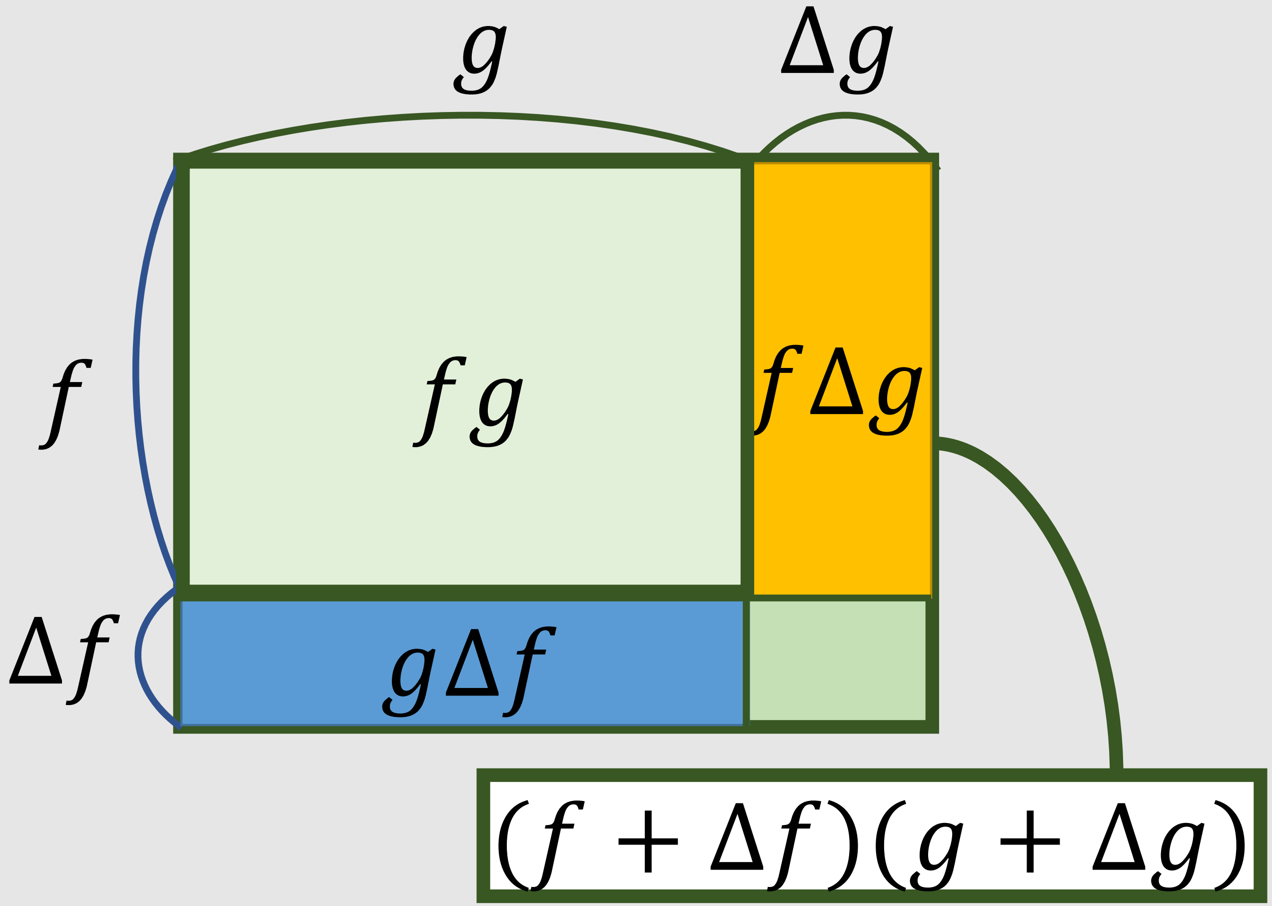
Vector Differentiation

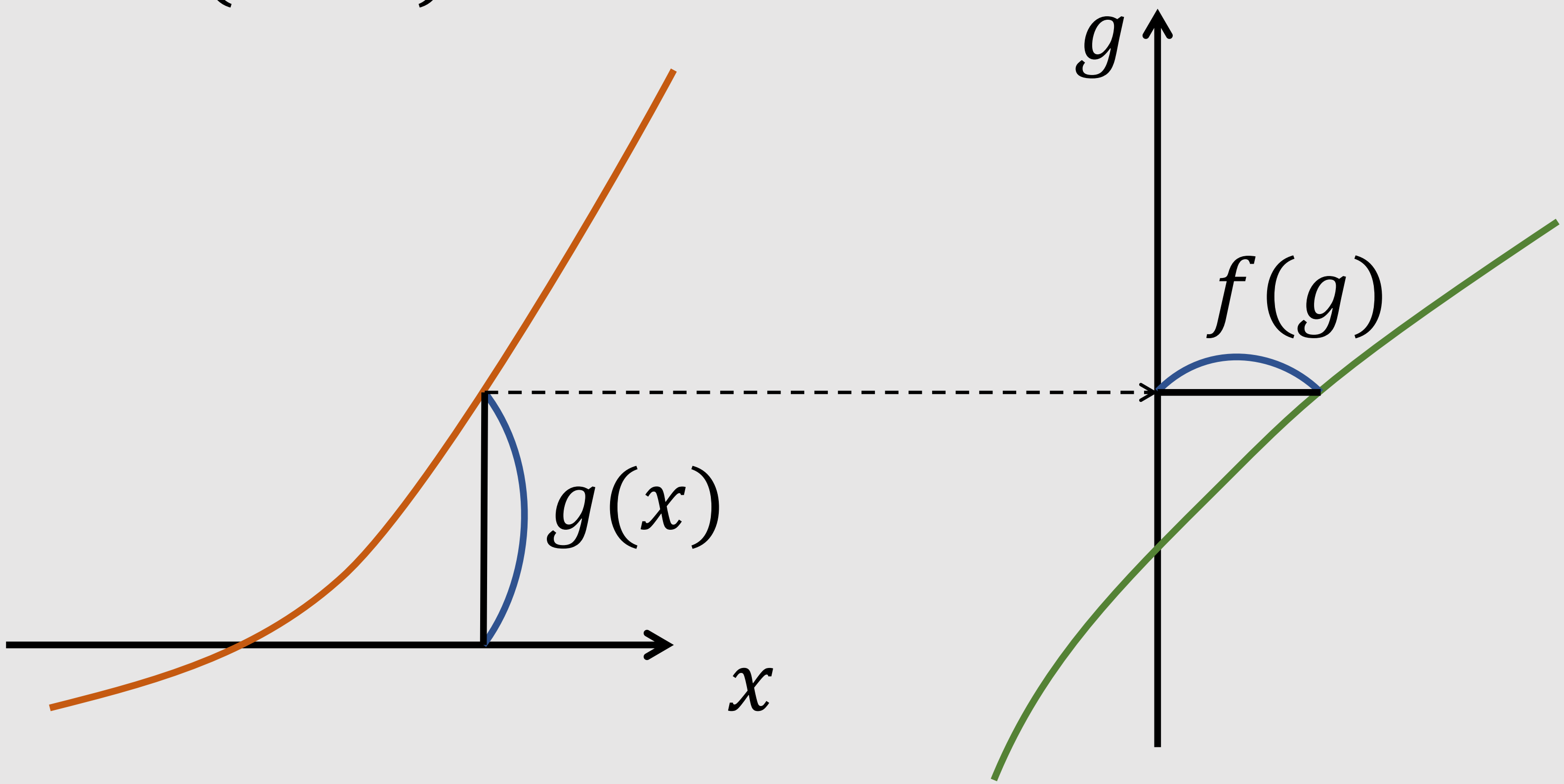
Review of Differentiation Rules: Multiple

$$(fg)' = f'g + fg'$$



Review of Differentiation Rules: Composite

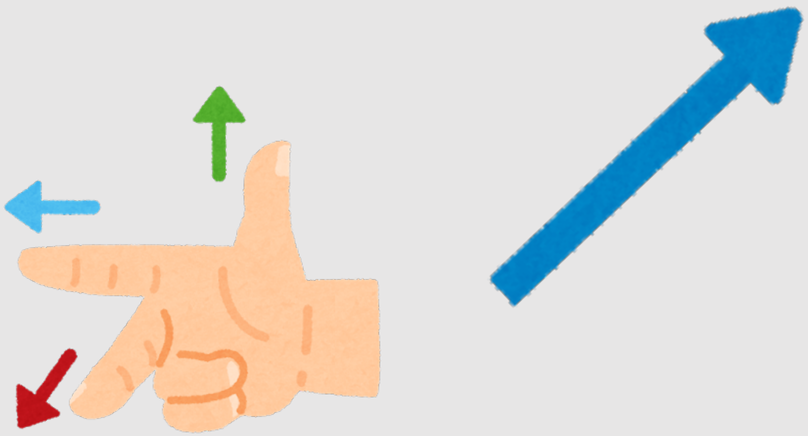
$$f'(g(x)) = f'\big(g(x)\big)g'(x)$$



Two Ways to Understand Vector

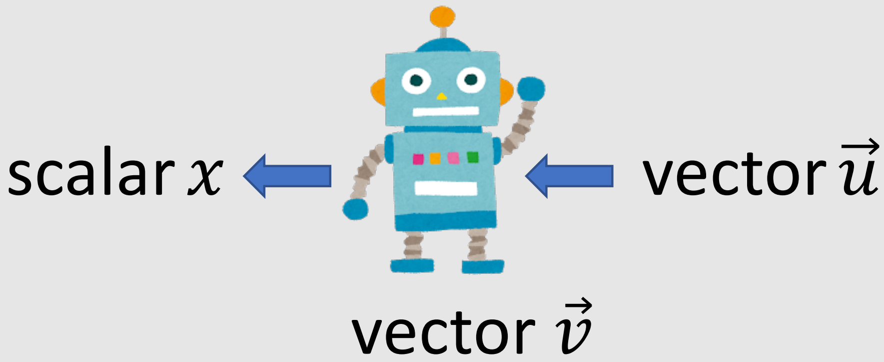
- Dot product define linear relationship between input & output

Spatial direction or position



Linear form

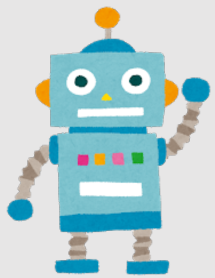
$$x = \vec{v} \cdot \vec{u}$$



Differential of a Scalar Function w.r.t. Vector

- Inner product with $d\vec{p}$ gives difference dW

$$dW = \frac{\partial W(\vec{p})}{\partial \vec{p}} \cdot d\vec{p}$$



vector

Since $d\vec{p}$ and dW change linearly, $\partial W(\vec{p})/\partial \vec{p}$ should be a vector

Differential of a Scalar Function w.r.t. Vector

- The differential can be written as

$$\frac{\partial W(\vec{p})}{\partial \vec{p}} = \frac{\partial W}{\partial p_1}\vec{e}_1 + \frac{\partial W}{\partial p_2}\vec{e}_2 + \frac{\partial W}{\partial p_3}\vec{e}_3$$

Let's Practice Differentiation!

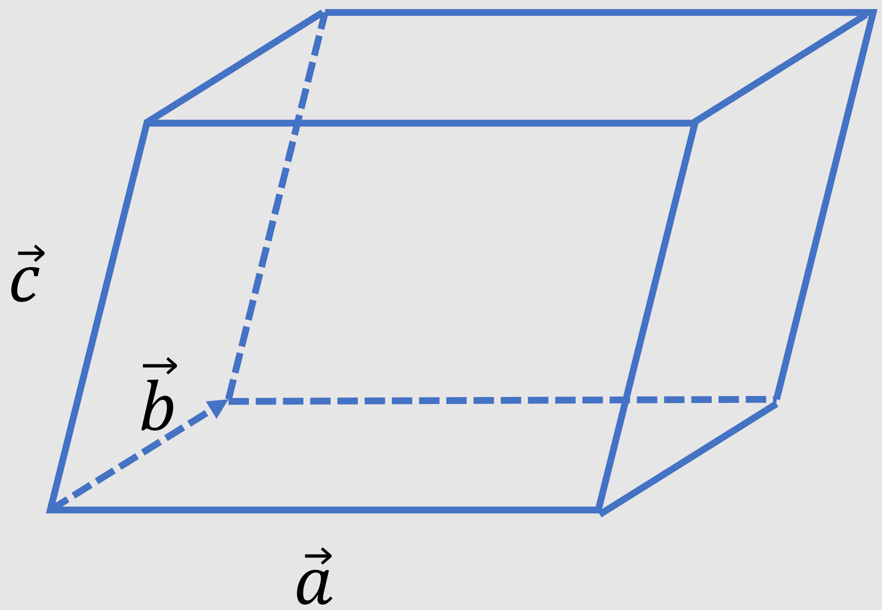
$$\mathrm{W}(\vec{p}) = \vec{a} \cdot \vec{p}$$

$$W(\vec{p}) = |\vec{p}|^2$$

$$W(\vec{p}) = {}^{1}/_{|\vec{p}|}$$

Scalar Triple Product (スカラー三重積)

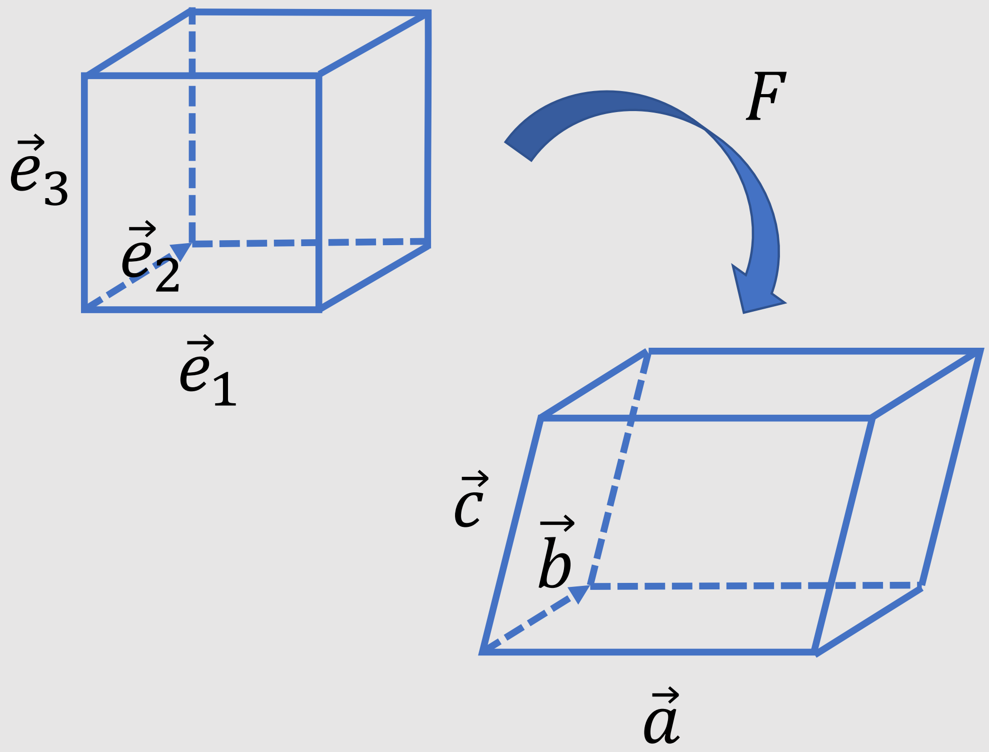
- Volume of parallelepiped



$$W = \vec{a} \cdot (\vec{b} \times \vec{c})$$

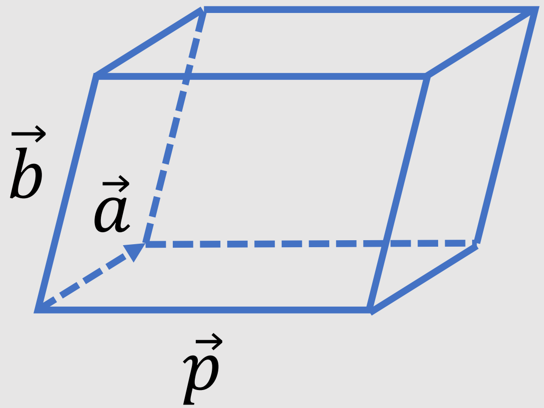
Scalar Triple Product and Determinant

- Scalar triple product is related to the volume change ratio



$$\vec{a}\cdot\left(\vec{b}\times\vec{c}\right) = det\underbrace{\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}}_{F}$$

Differential of Scalar Triple Product



$$W(\vec{p}) = \vec{b} \cdot (\vec{a} \times \vec{p})$$

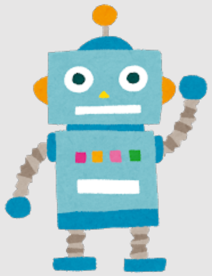
$$\frac{\partial W}{\partial \vec{p}} = ?$$

check it out!

Differentiation of Vector w.r.t. Vector

- Inner product with $d\vec{p}$ gives difference $d\vec{F}$

$$d\vec{F} = \frac{\partial \vec{F}(\vec{p})}{\partial \vec{p}} \cdot d\vec{p}$$



matrix

Since $d\vec{F}$ and $d\vec{p}$ change linearly, $\partial \vec{F}(\vec{p})/\partial \vec{p}$ should be a matrix

Differentiation of Vector w.r.t. Vector

- Inner product with $d\vec{p}$ gives difference $d\vec{F}$

$$\frac{\partial \vec{F}}{\partial \vec{p}} = \frac{\partial F_1}{\partial p_1}\vec{e}_1 \otimes \vec{e}_1 + \frac{\partial F_1}{\partial p_2}\vec{e}_1 \otimes \vec{e}_2 + \frac{\partial F_2}{\partial p_1}\vec{e}_2 \otimes \vec{e}_1 \cdots$$

$$= \sum_i \sum_j \frac{\partial F_i}{\partial p_j}\vec{e}_i \otimes \vec{e}_j$$

Let's Practice Differentiation!

$$\vec{F} = A \cdot \vec{p}$$

$$\vec{F} = \vec{p}$$

$$\vec{F} = (\vec{a} \cdot \vec{p})\vec{b}$$

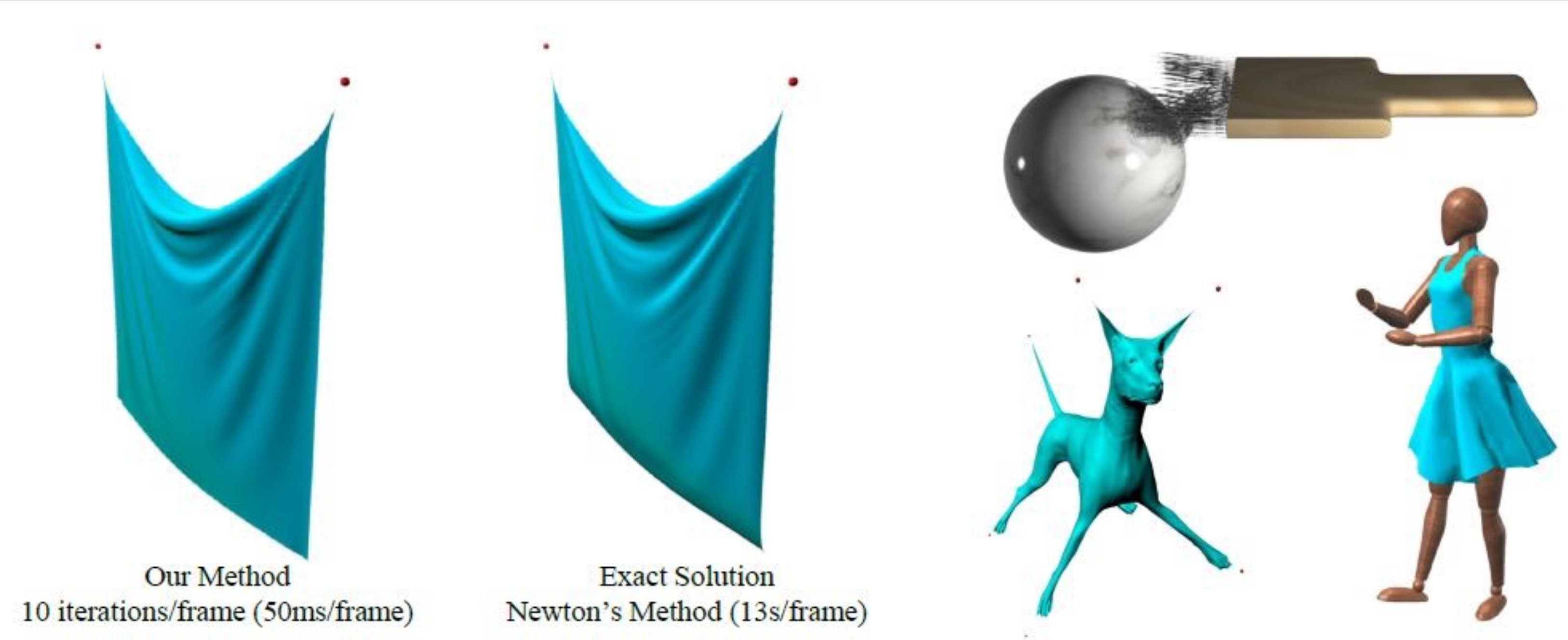
$$\vec{F} = \vec{p}/\|\vec{p}\|$$

check it out!

Mass-Spring System

バネ・質点モデル

Widely Used Model for Elastic Objects

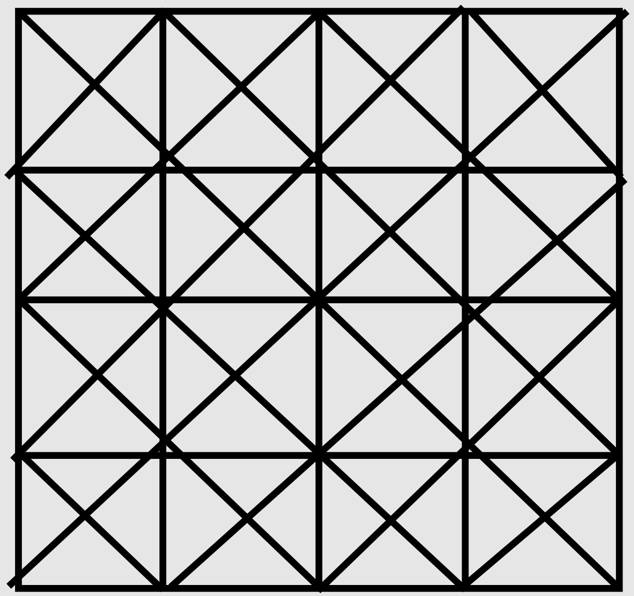


Tiantian Liu, Adam W. Bargteil, James F. O'Brien, Ladislav Kavan. **Fast Simulation of Mass-Spring Systems**. *ACM Transaction on Graphics 32(6) [Proceedings of SIGGRAPH Asia], 2013.*

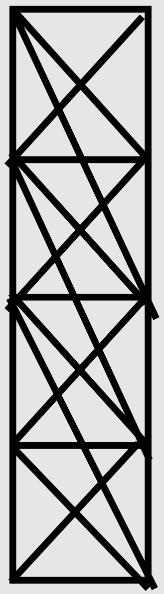
Modeling Cloth, Rod, and Solid

- Heuristic layout of springs to prevent undesirable deformation

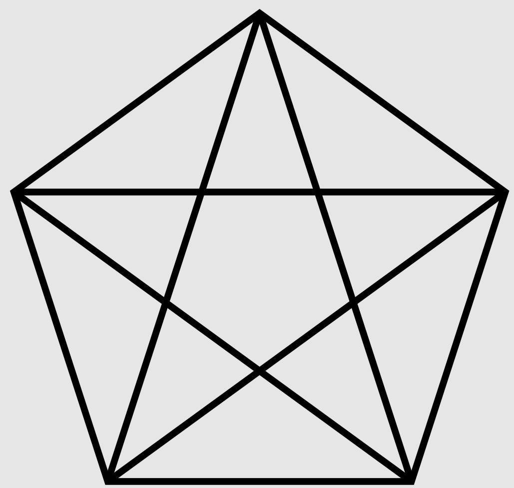
Cloth



Rod

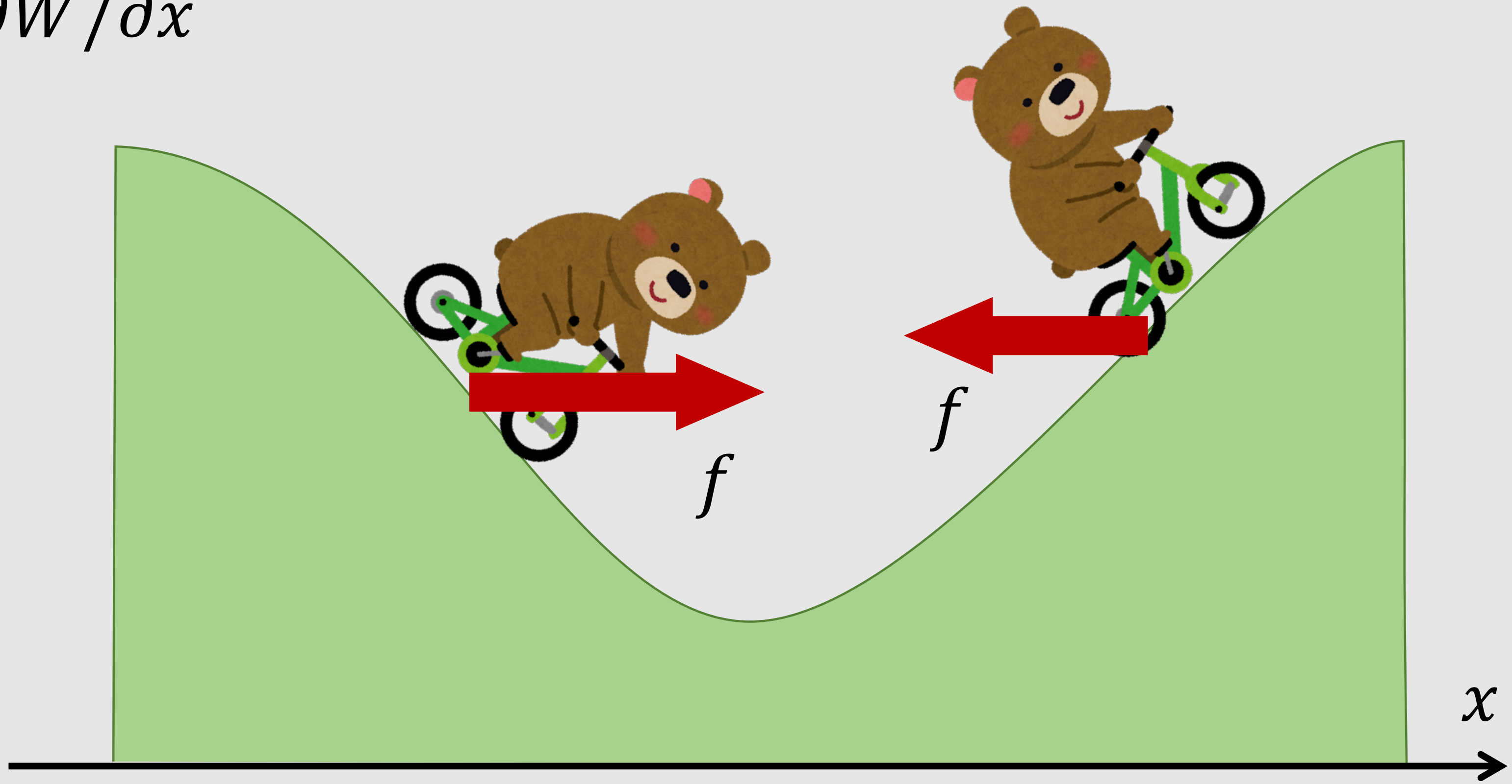


Solid



Potential Energy: Energy Given by Position

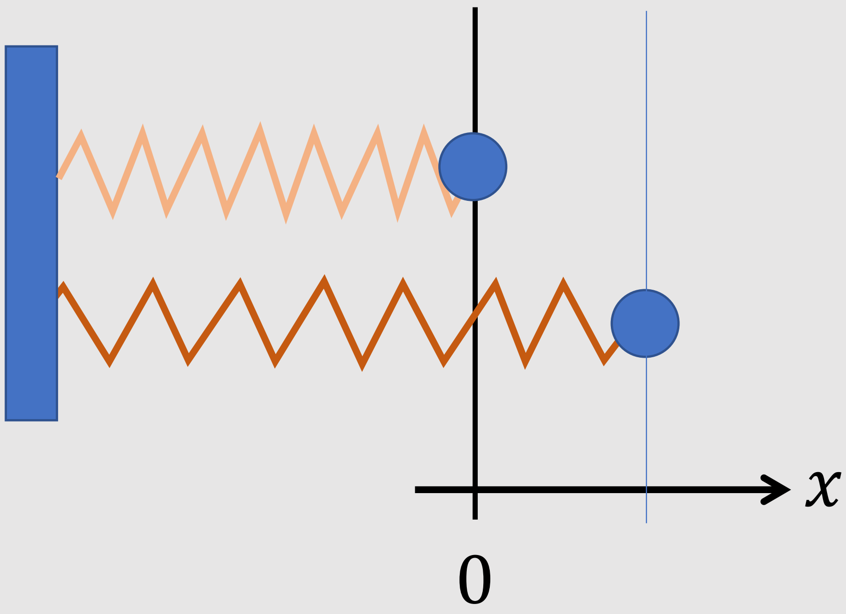
- Gravitational potential energy: $W = -mgh$
- Force: $f = -\partial W / \partial x$



Hooke's Law

- Force changes linearly to the displacement

L: rest length



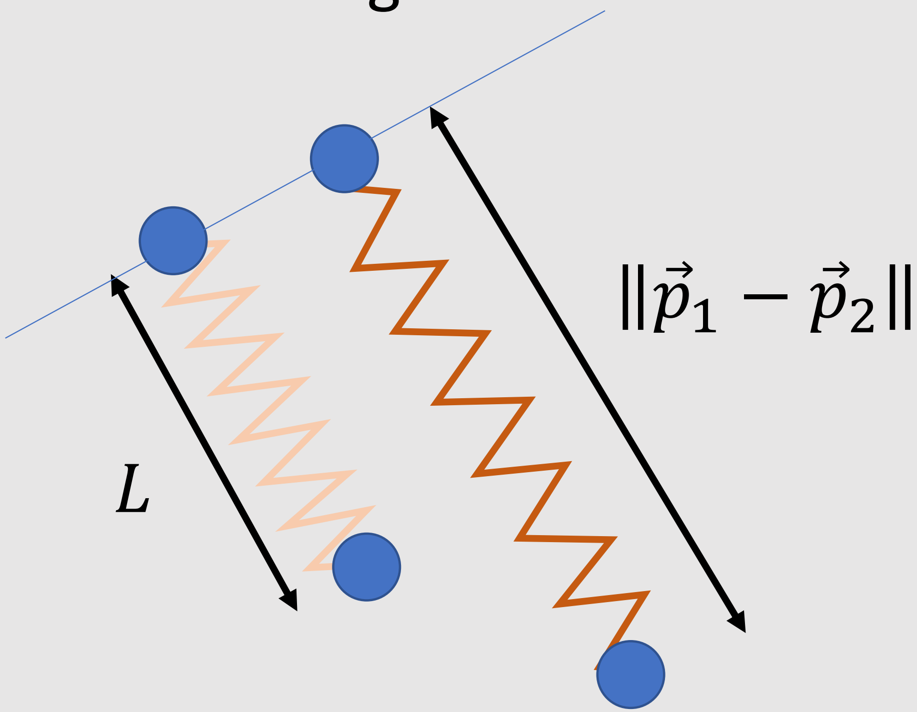
$$f = -kx$$

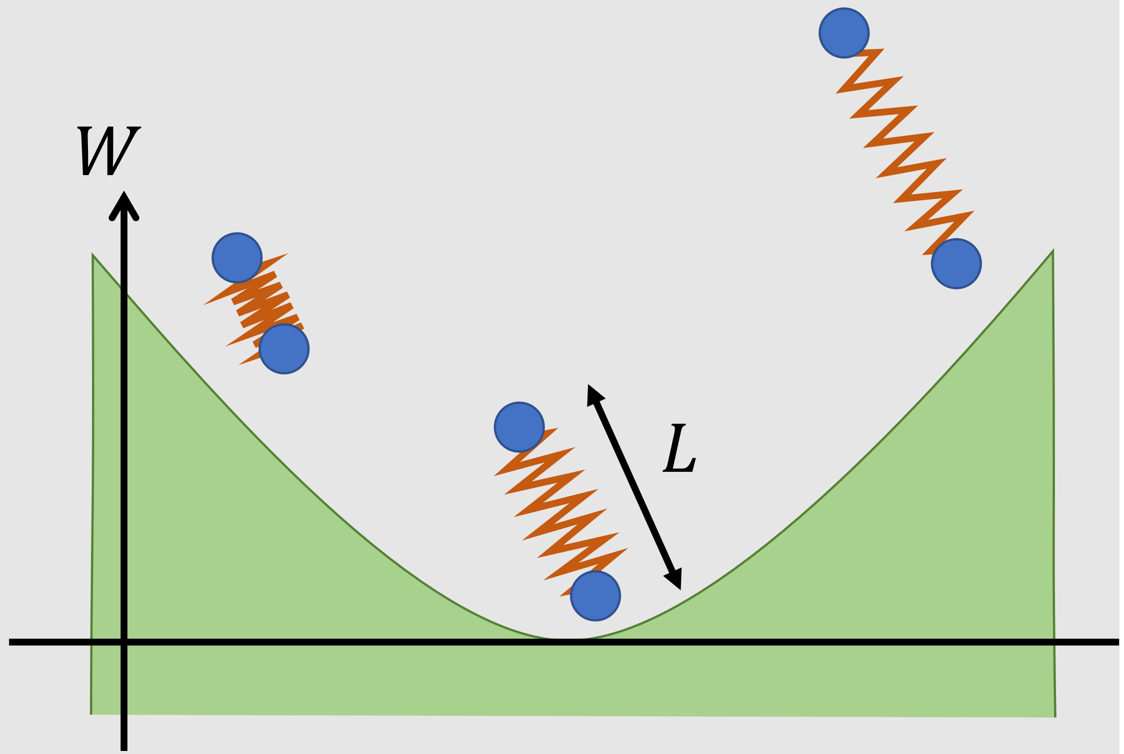
$$\mathrm{W} = \int_0^x f\,dx = ?$$

A Spring in 3D

$$W(\vec{p}_1, \vec{p}_2) = \frac{1}{2}k(\|\vec{p}_1 - \vec{p}_2\| - L)^2$$

L: rest length



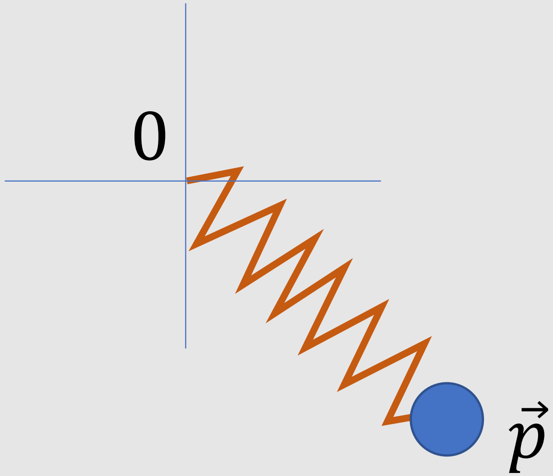


Force of the 3D Spring

- One end is fixed to the origin

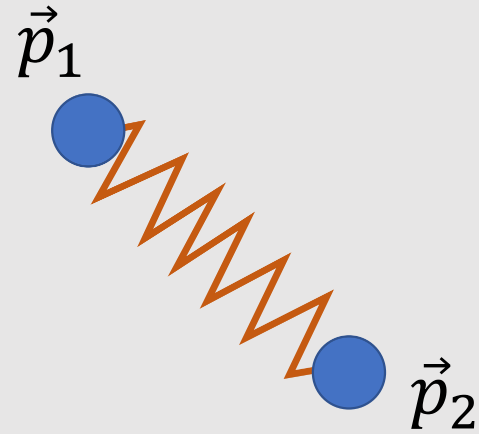
$$W(\vec{p}) = \frac{1}{2}k(\|\vec{p}\| - L)^2$$

$$f = \frac{\partial W}{\partial \vec{p}} = ?$$



Force of the 3D Spring

- Both ends are free



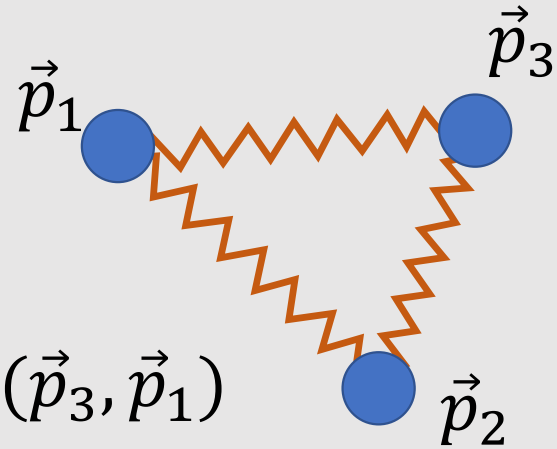
$$W(\vec{p}_1, \vec{p}_2) = \frac{1}{2} k (\|\vec{p}_1 - \vec{p}_2\| - L)^2$$

$$f_1 = \frac{\partial W}{\partial \vec{p}_1} = ?$$

$$f_2 = \frac{\partial W}{\partial \vec{p}_2} = ?$$

check it out!

Three Springs



- Summing up three energy terms

$$W_{total}(\vec{p}_1, \vec{p}_2, \vec{p}_3) = W(\vec{p}_1, \vec{p}_2) + W(\vec{p}_2, \vec{p}_3) + W(\vec{p}_3, \vec{p}_1)$$

$$f_1 = \frac{\partial W_{total}}{\partial \vec{p}_1} = ?$$

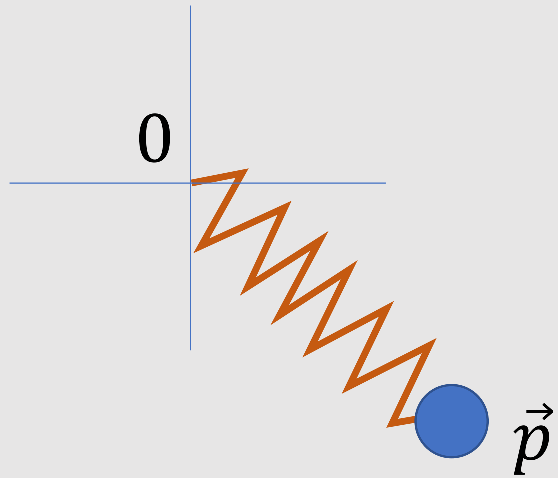
$$f_2 = \frac{\partial W_{total}}{\partial \vec{p}_2} = ?$$

$$f_3 = \frac{\partial W_{total}}{\partial \vec{p}_3} = ?$$

check it out!

Hessian of Elastic Potential Energy

- One end is fixed to the origin



$$W(\vec{p}) = \frac{1}{2}k(\|\vec{p}\| - L)^2$$

$$\frac{\partial W}{\partial \vec{p}} = ?$$