

Optimization with Constraints

Why Constraints?

- Solid deformation
 - Non penetration constraints



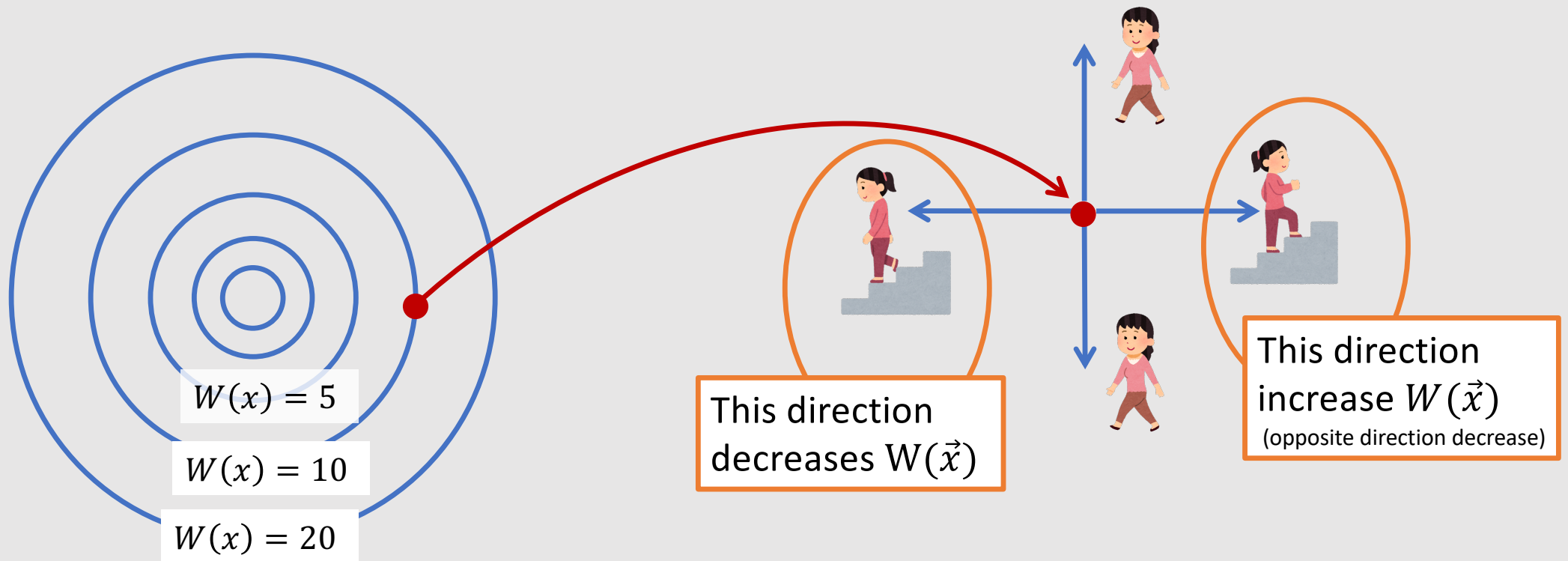
Credit: Damnsoft 09 @ Wikipedia

- Fluid
 - incompressibility constraints: vortex



Credit: Astrobob @ Wikipedia

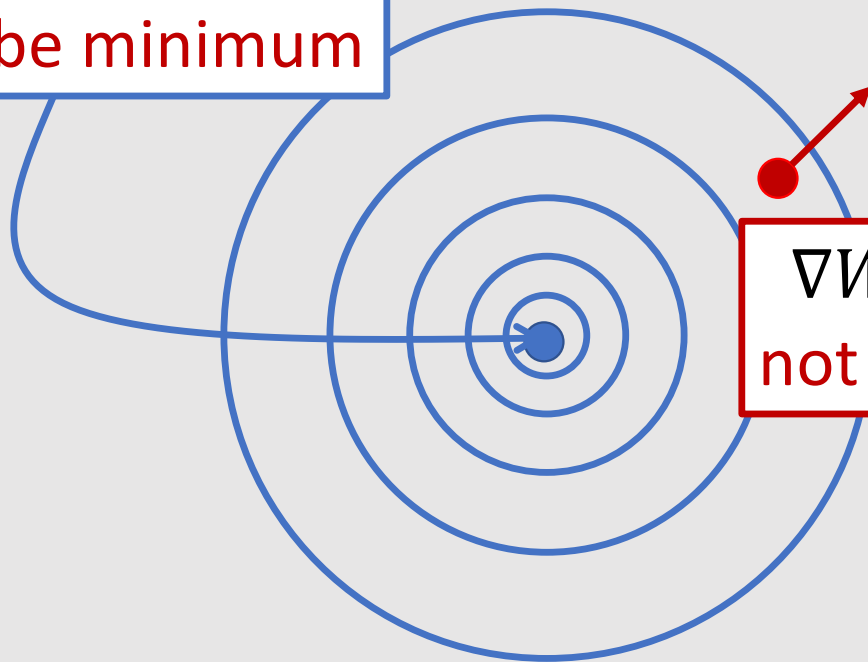
Not Minimum If Its Gradient is not Zero



Maybe Minimum if Gradient is Zero

- Find a candidate where the **gradient is zero** $\nabla W(\vec{x}) = 0$

$\nabla W(\vec{x}) = 0$
maybe minimum



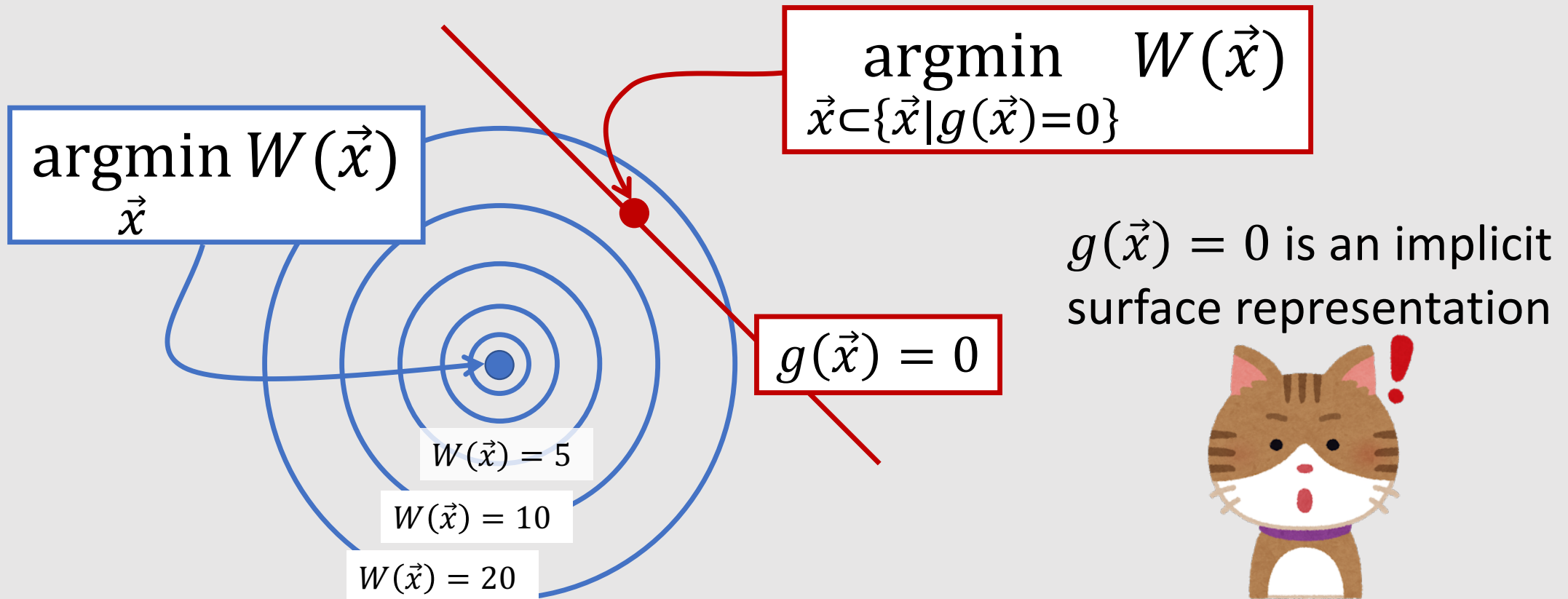
$\nabla W(\vec{x}) \neq 0$
not minimum

find the **root** of gradient!

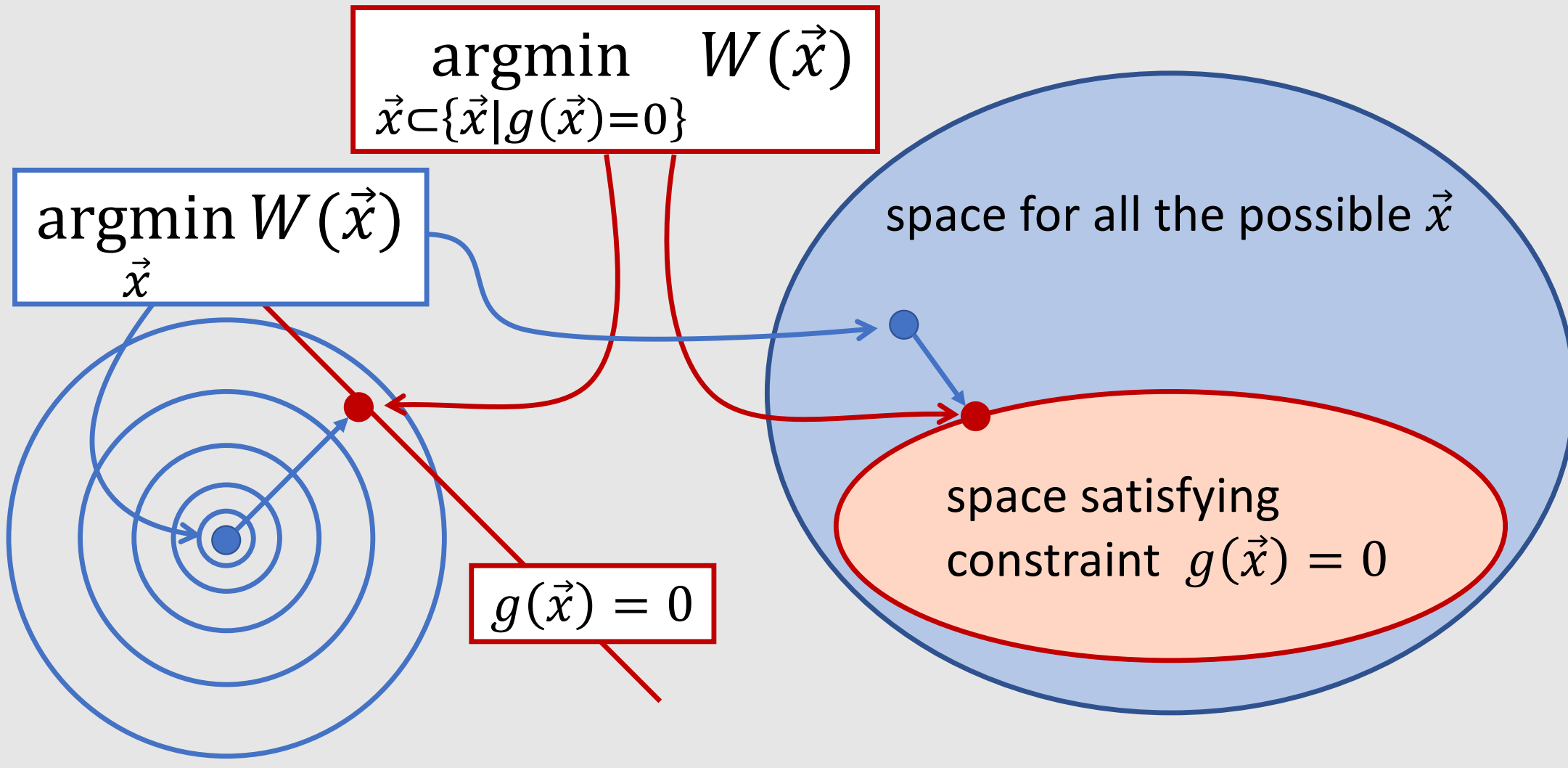


Optimization with Constraint

- Find a point $\vec{x}$ where the function $W(\vec{x})$ is minimized **while** satisfying $g(\vec{x}) = 0$



Abstract View of the Solution Space



Three Approaches to Handle Constraints

- ***Degree of Freedom (DoF) elimination***

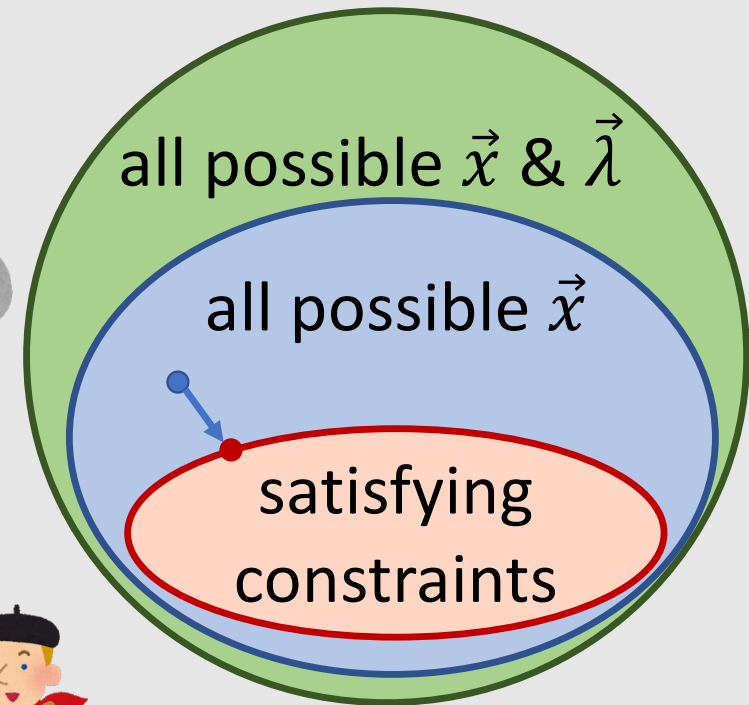
- Optimization in the constraint space
- Find minimum ● in 

- ***Penalty method***

- Approximate constraint as energy
- Find minimum ● in 

- ***Lagrange multiplier method***

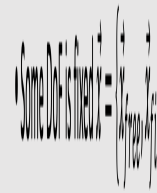
- Chose gradient parallel to the constraint's gradient
- Find **extremum** ● in 



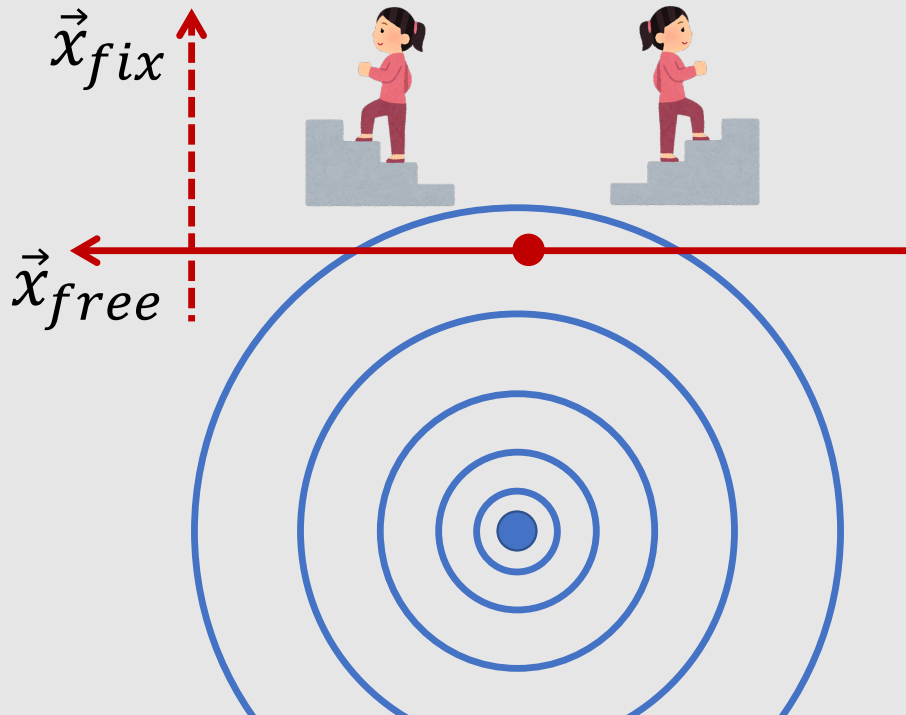
Degree of Freedom (DoF) Elimination

Degree of Freedom (DoF) Elimination

- Some DoF is fixed $\vec{x} = \{\vec{x}_{free}, \vec{x}_{fix}\}$



walk only on the
line to find the
minimum



$$dW(\vec{x}_{free}, \vec{x}_{fix}) = 0$$

$$\nabla W \cdot \begin{pmatrix} d\vec{x}_{free} \\ d\vec{x}_{fix} \end{pmatrix} = 0$$

$= 0$

Newton Method for DoF Elimination



- Update, Gradient and Hessian for Free/Fix DoF

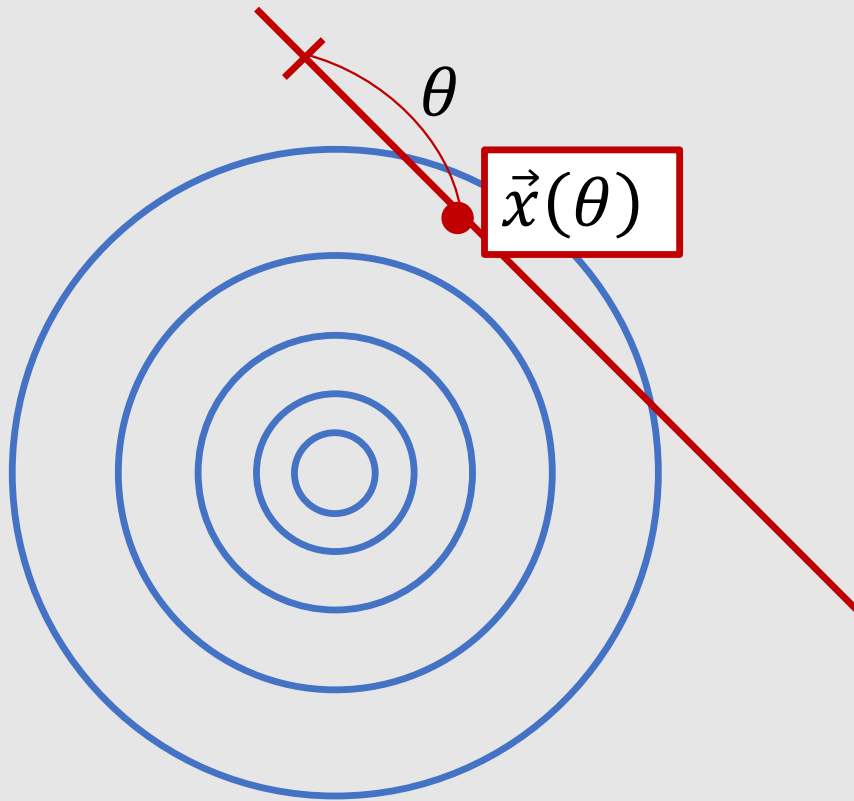
$$\nabla W = \begin{pmatrix} \nabla W_{free} \\ \nabla W_{fix} \end{pmatrix} \quad \nabla^2 W = \begin{bmatrix} \nabla^2 W_{free,free} & \nabla^2 W_{free,fix} \\ \nabla^2 W_{fix,free} & \nabla^2 W_{fix,fix} \end{bmatrix}$$

- Update only $d\vec{x}_{free}$ (while $d\vec{x}_{fix} = 0$) to achieve $\nabla W_{free} = 0$

$$d\vec{x} = \begin{pmatrix} d\vec{x}_{free} \\ d\vec{x}_{fix} \end{pmatrix} = \begin{bmatrix} \nabla^2 W_{free,free} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}^{-1} \begin{pmatrix} \nabla W_{free} \\ \mathbf{0} \end{pmatrix}$$

DoF Elimination for General Constraint

Parameterize solution $\vec{x}(\theta)$ such that constraints naturally satisfy



$$\operatorname{argmin}_{\vec{x} \in \{\vec{x} | g(\vec{x}) = 0\}} W(\vec{x}) \quad \longrightarrow \quad \operatorname{argmin}_{\theta} W(\vec{x}(\theta))$$

e.g., $g(\vec{x}) = x + y + 2 = 0$

$$\longrightarrow \begin{cases} x = +\theta - 1 \\ y = -\theta - 1 \end{cases}$$

Minimize Parameterized Solution

$$\underset{\theta}{\operatorname{argmin}} W(\vec{x}(\theta))$$



Newton-Raphson method

$$d\theta = - \left[\frac{\partial^2 W}{\partial \theta^2} \right]^{-1} \left(\frac{\partial W}{\partial \theta} \right)$$



find the **root** of gradient!



$$\frac{\partial W}{\partial \theta} = \frac{\partial W}{\partial \vec{x}} \frac{\partial \vec{x}}{\partial \theta} \quad \text{and} \quad \frac{\partial^2 W}{\partial \theta^2} = \left(\frac{\partial \vec{x}}{\partial \theta} \right)^T \frac{\partial^2 W}{\partial \vec{x}^2} \left(\frac{\partial \vec{x}}{\partial \theta} \right)$$

DoF Elimination Use Cases

Fixing deformation in XYZ direction



Optimization for rotation



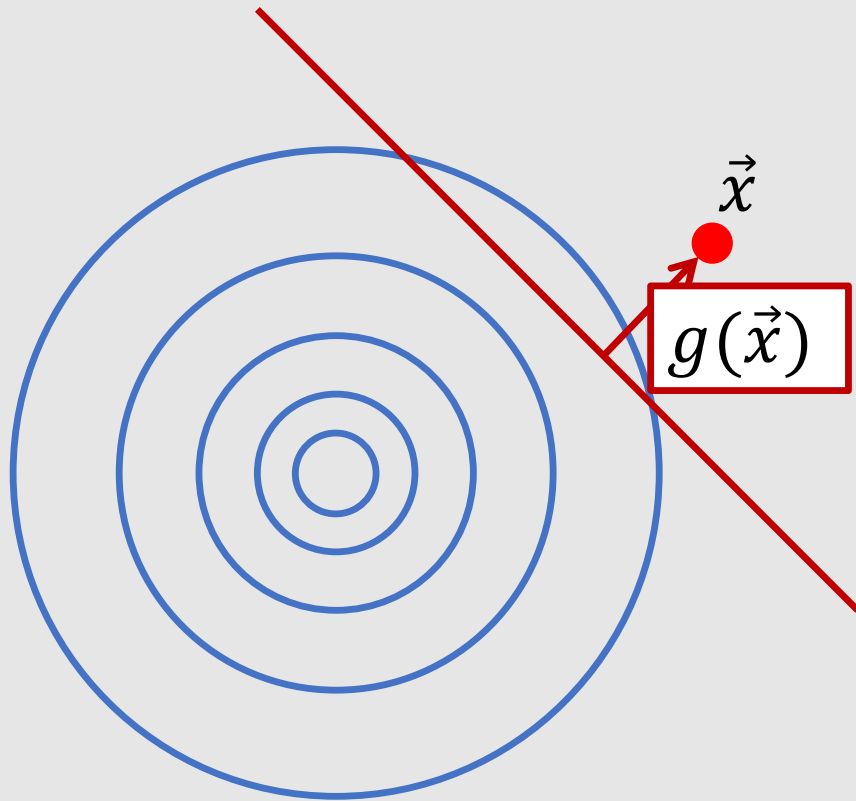
Vortex method for fluid simulation



Penalty Method (Soft Constraint)

ペナルティー法

Deviation from Constraints is $g(\vec{x})$

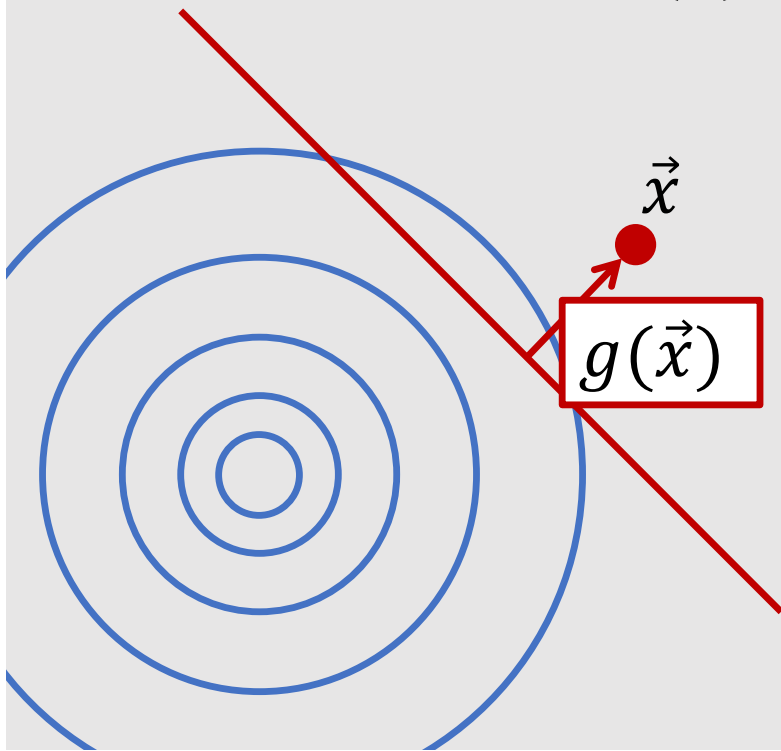


Let's put a penalty on the deviation from the constraint

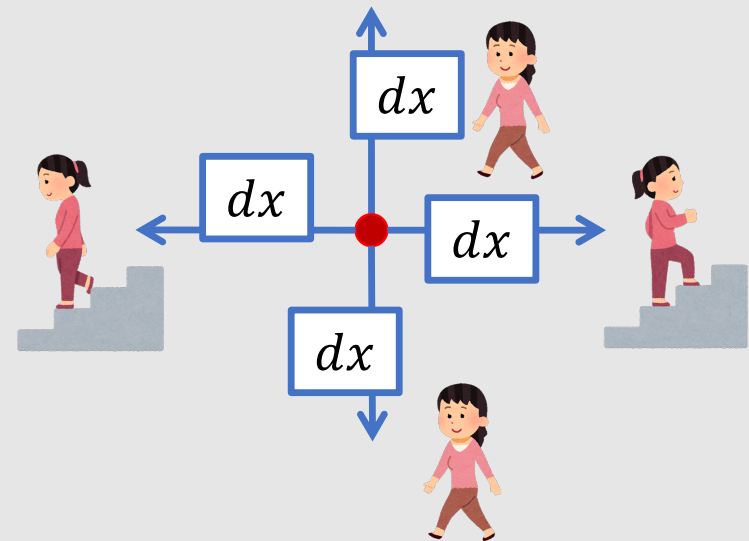
Penalty Method: Constraint as Energy

- Adding additional energy to encourage constraint

minimize $W(\vec{x}) + \alpha g^2(\vec{x}) \Rightarrow$ If α is large, $g(\vec{x})$ becomes small



$d\vec{x}$ can be **all the direction**



Linear System for Penalty Method

$$\operatorname{argmin}_{\vec{x}} W(\vec{x}) + \alpha g^2(\vec{x})$$

Minimize $W + g$ with Newton's method:

$$d\vec{x} = -[\underbrace{\nabla^2 W + \alpha \nabla^2 g^2}_{2\nabla g \cdot \nabla g + 2\nabla^2 g}]^{-1}(\underbrace{\nabla W + \alpha \nabla g^2}_{2g\nabla g})$$



find the **root** of gradient!



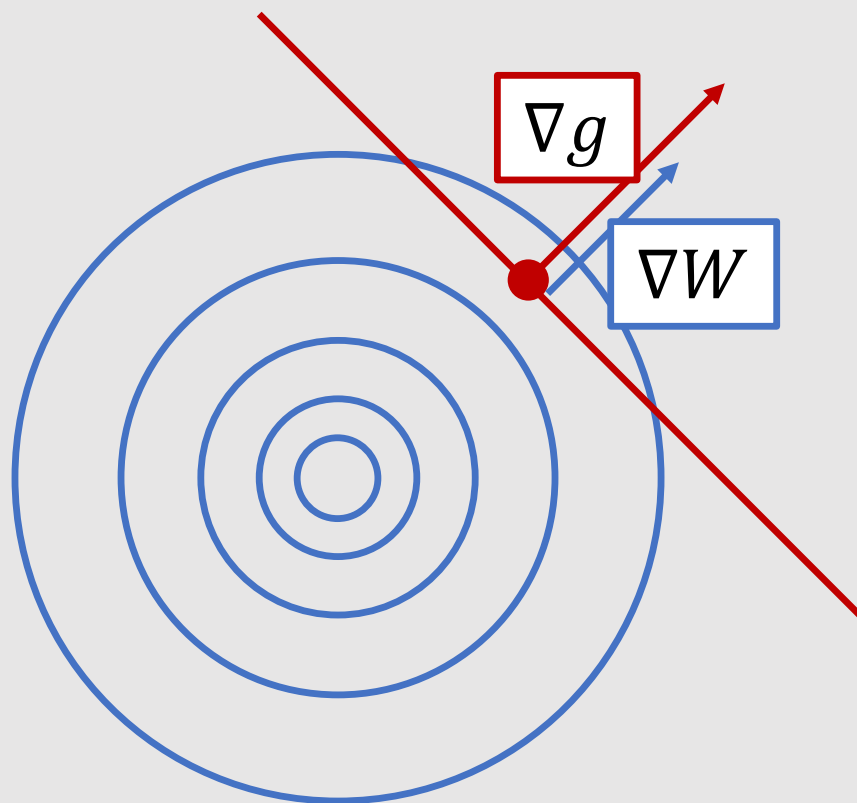
Lagrange Multiplier Method

ラグランジュ未定乗数法

je ne sais quoi!

Lagrange Multiplier Method

- At minimum point, two gradients ∇W , ∇g should be parallel



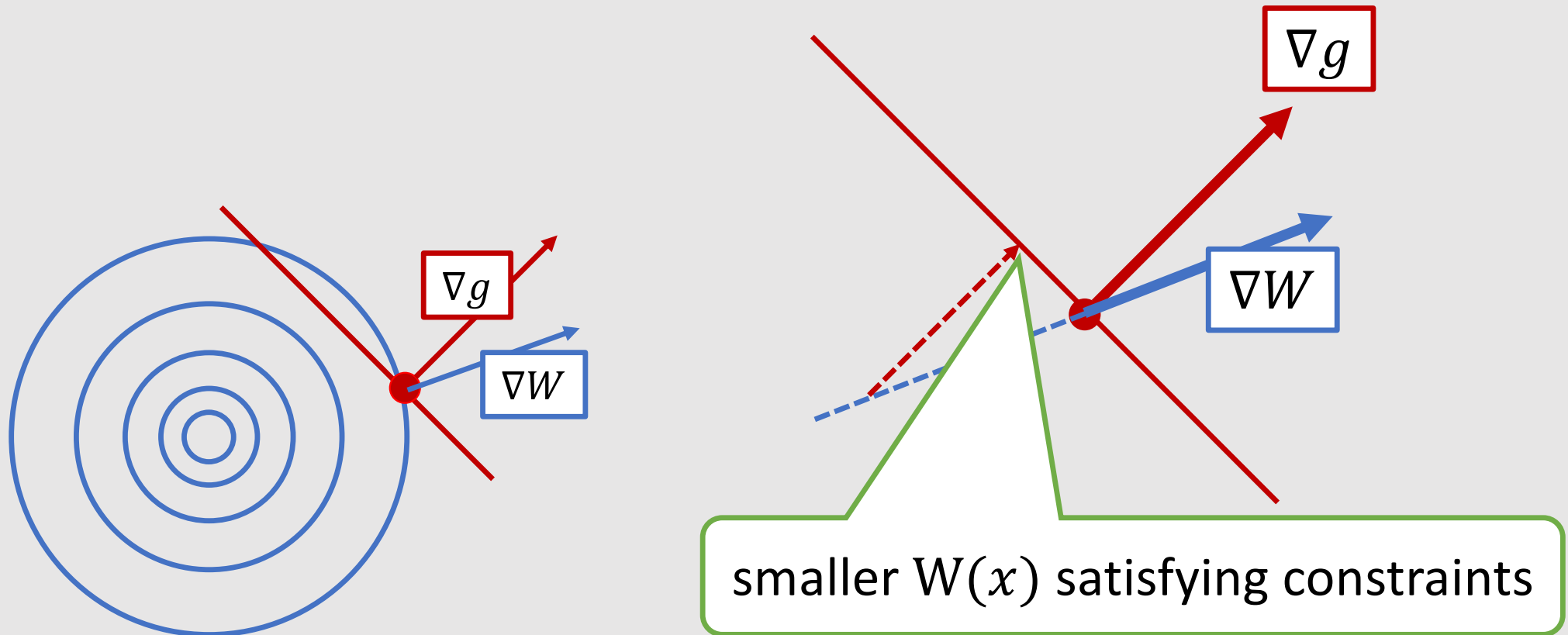
$$\nabla W \parallel \nabla g$$



$$\exists \lambda \neq 0 \text{ s.t. } \nabla W = \lambda \nabla g$$

Why Parallel at Constrained Minimum?

- If $\nabla W, \nabla g$ are **not parallel**, smaller $W(x)$ exists satisfying constraints



Find **Saddle Point** not Minima for LM Method

- We changed minimization problem to **saddle point finding problem**

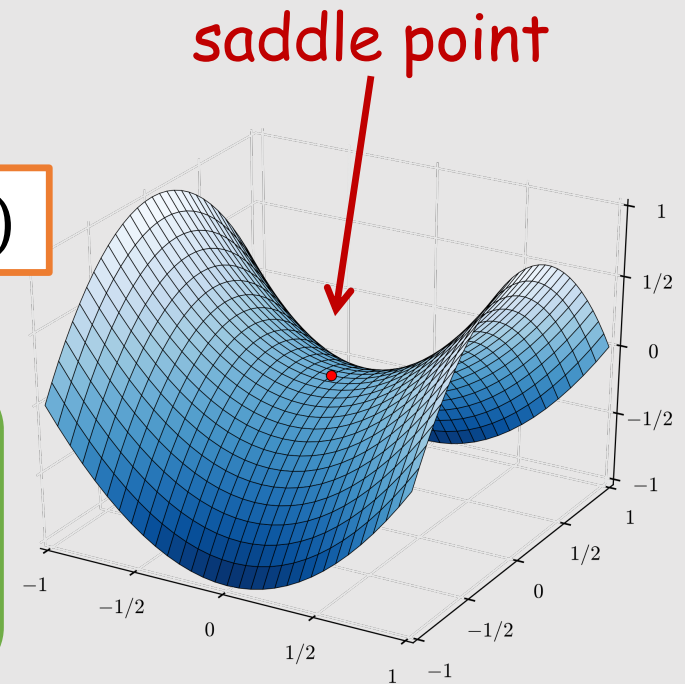
$$\nabla W(\vec{x}) = \lambda \nabla g(\vec{x})$$



$$\nabla \bar{W}(\vec{x}, \lambda) = 0 \text{ where } \bar{W}(\vec{x}, \lambda) = W(\vec{x}) - \lambda g(\vec{x})$$



Don't minimize $\bar{W}(\vec{x}, \lambda)$. Find where the gradient is zero using the **Newton method**



Credit: Nicoguaro @ Wikipedia

Lin. System for Lagrange Multiplier Method

$$\begin{pmatrix} \nabla W(\vec{x}) - \lambda \nabla g(\vec{x}) \\ -g(\vec{x}) \end{pmatrix} = H(\vec{x}, \lambda) = 0$$

Newton-Raphson method

$$\begin{pmatrix} d\vec{x} \\ d\lambda \end{pmatrix} = -[\nabla H]^{-1} H$$
$$= - \begin{bmatrix} \nabla^2 W(\vec{x}) - \lambda \nabla^2 g(\vec{x}) & -\nabla g(\vec{x}) \\ -\nabla g(\vec{x}) & 0 \end{bmatrix} \begin{pmatrix} \nabla W(\vec{x}) - \lambda \nabla g(\vec{x}) \\ -g(\vec{x}) \end{pmatrix}$$

find the root!



Let's Practice Lagrange Multiplier Method

Maximize $f(x, y) = x + y$ where $g(x, y) = x^2 + y^2 - 1 = 0$

check it out!

