Angular Velocity

角速度

What is Skew Symmetric Matrix?

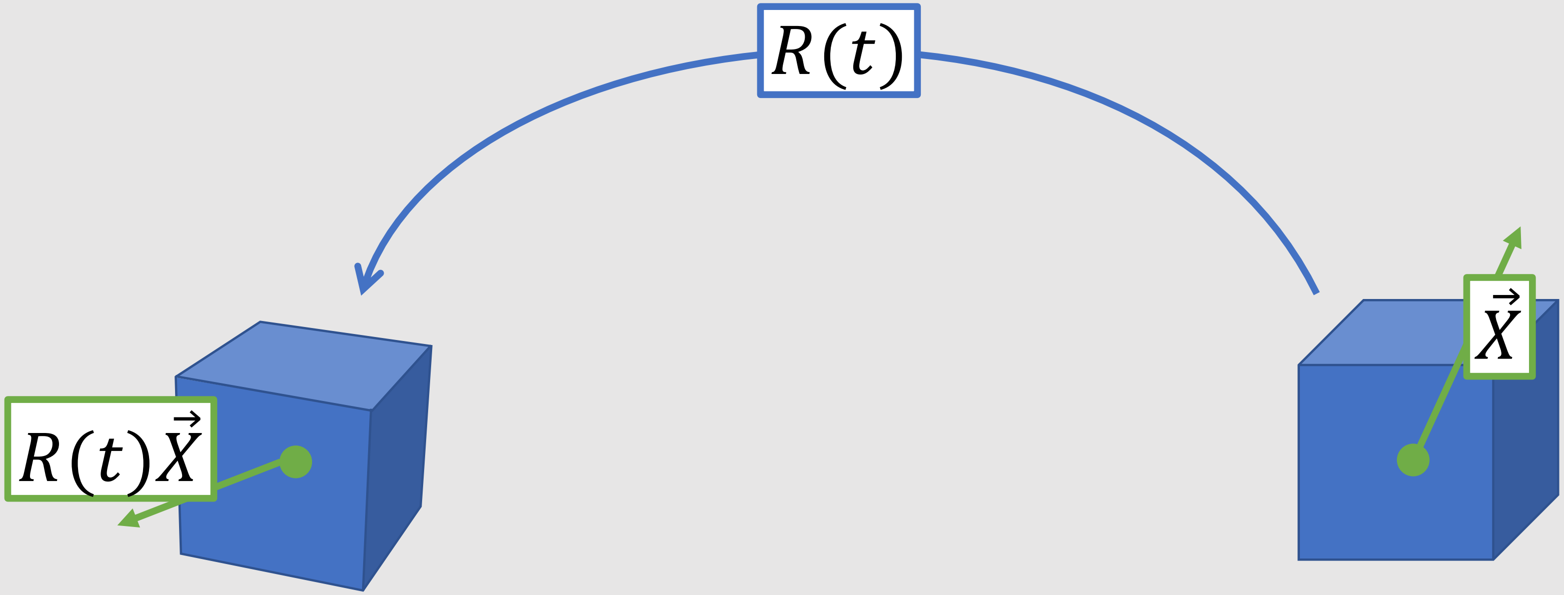
- 3x3 skew symmetric matrix represents a vector

$$A^T = -A$$

$$A = \text{Skew}(\vec{a}) = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

$$\text{Skew}(\vec{a})\,\vec{b} = \vec{a} \times \vec{b}$$

Reference & Current Configuration



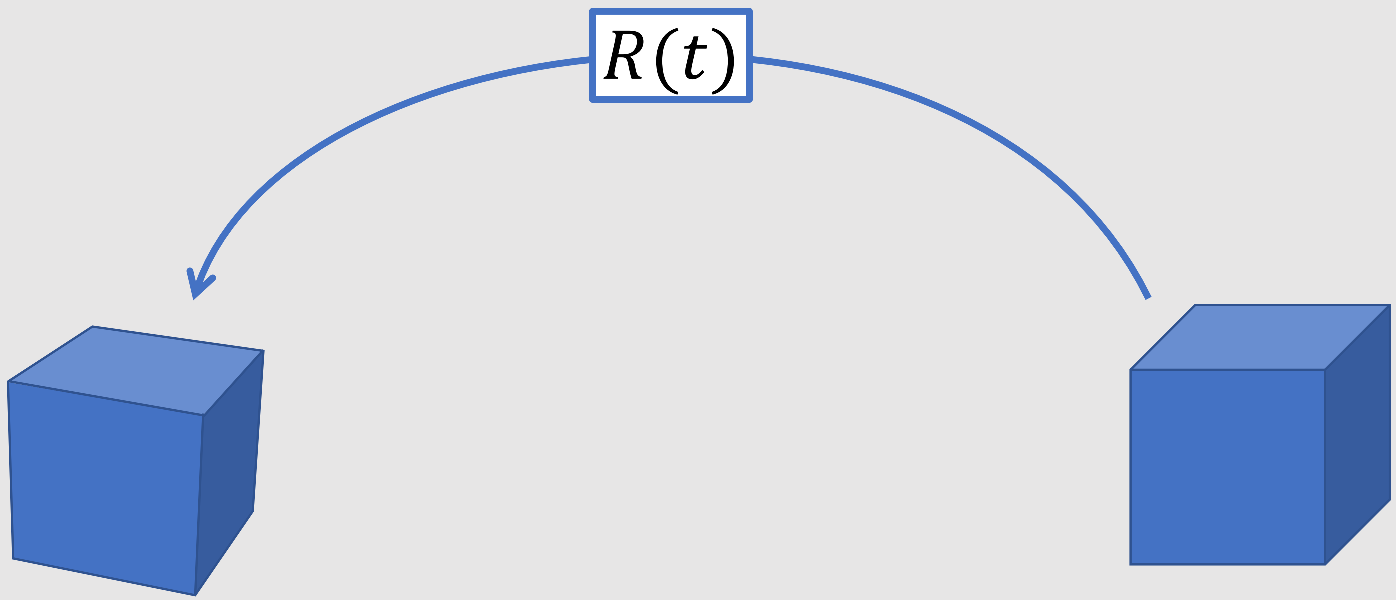
current configuration

reference configuration

Constraint on the Rotation

walking on the edge!

Rotation $R(t)$ changes under constraint $R(t)^T R(t) = I$



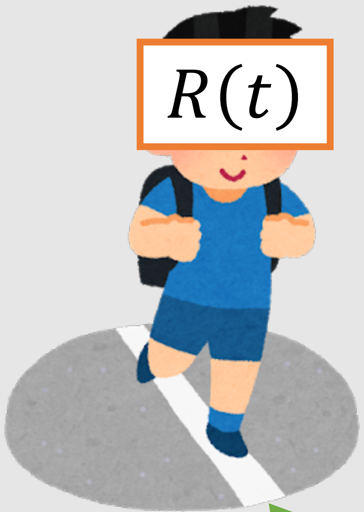
current configuration

reference configuration

Differentiation of Rotation Matrix

Continuous change under constraints $R(t)$ → DoF elimination

walking on the edge!



$$R(t)^T R(t) = I$$

Differentiation of Rotation Matrix

Continuous change under constraints $R(t)$ → DoF elimination

$$\frac{d}{dt}(RR^T) = 0$$

↓ $\dot{R}R^T + R\dot{R}^T = 0$

$$\dot{R}R^T + \left(\dot{R}R^T\right)^T = 0$$

$\dot{R}R^T$ is skew-symmetric

$$\dot{R}R^T = \mathrm{Skew}(\vec{\omega})$$

$$\dot{R} = \mathrm{Skew}(\vec{\omega})R$$

Another Differentiation of Rotation Matrix

Continuous change under constraints $R(t) \rightarrow$ DoF elimination

$$\frac{d}{dt}(RR^T) = 0$$

$\dot{R}R^T + R\dot{R}^T = 0$

$$\dot{R}R^T + \left(\dot{R}R^T\right)^T = 0$$

$\dot{R}R^T$ is skew-symmetric

$$\dot{R}R^T = \mathrm{Skew}(\vec{\omega})$$

$$\dot{R} = \mathrm{Skew}(\vec{\omega})R$$

$$\frac{d}{dt}(R^T R) = 0$$

$\dot{R}^T R + R\dot{R} = 0$

$$\left(R\dot{R}\right)^T + R\dot{R} = 0$$

$R\dot{R}$ is skew-symmetric

$$R^T\dot{R} = \mathrm{Skew}(\vec{\Omega})$$

$$\dot{R} = R\mathrm{Skew}(\vec{\Omega})$$

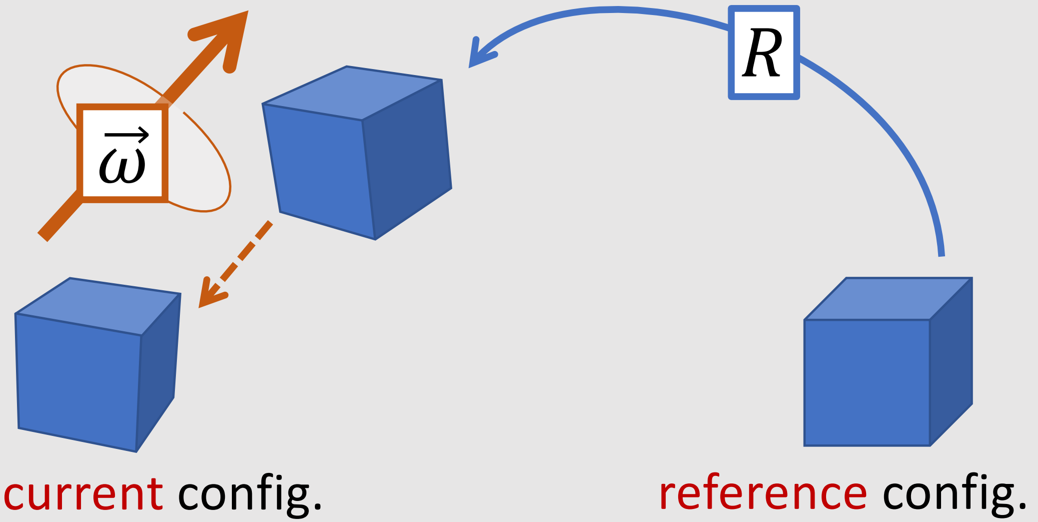
Angular Velocities are Vector Values

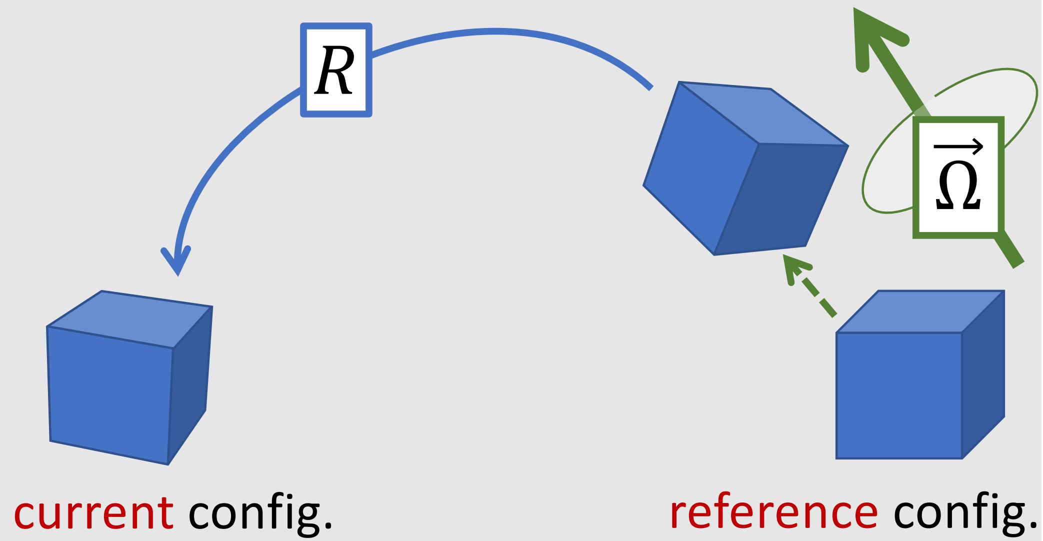
$\vec{\omega}$: velocity in current config.

$$\dot{R} = \text{Skew}(\vec{\omega})R$$

$\vec{\Omega}$: velocity in reference config.

$$\dot{R} = R\ \text{Skew}(\vec{\Omega})$$





Relationship between Two Angular Velocities

$\vec{\omega}$: velocity in current config.

$$\dot{R} = \text{Skew}(\vec{\omega})R$$

$\vec{\Omega}$: velocity in reference config.

$$\dot{R} = R\ \text{Skew}(\vec{\Omega})$$

$$\text{Skew}(\vec{\omega}) = R\ \text{Skew}\left(\vec{\Omega}\right)R^T$$

$$\vec{\omega} = R\ \vec{\Omega}$$

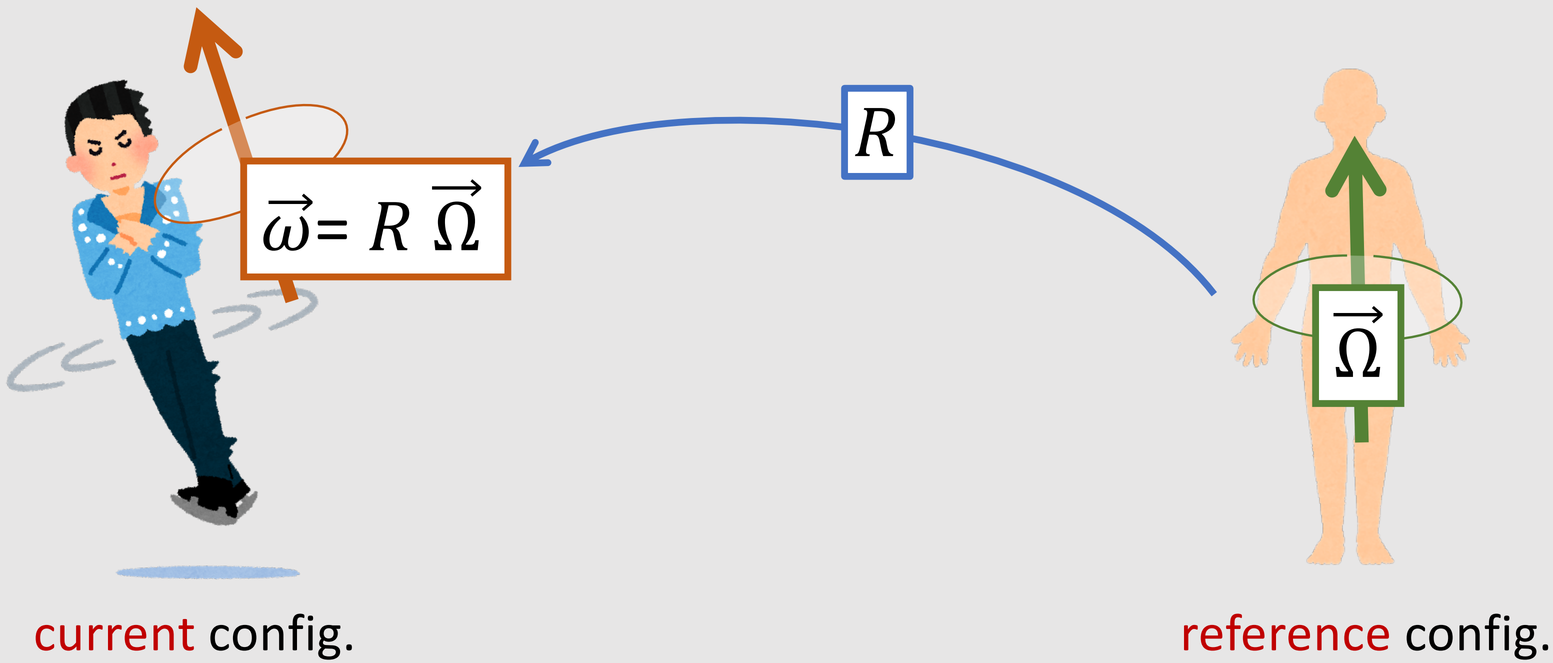
Angular Velocities are Vector Values

$\vec{\omega}$: velocity in current config.

$$\dot{R} = \mathrm{Skew}(\vec{\omega})R$$

$\vec{\Omega}$: velocity in reference config.

$$\dot{R} = R\,\mathrm{Skew}(\vec{\Omega})$$

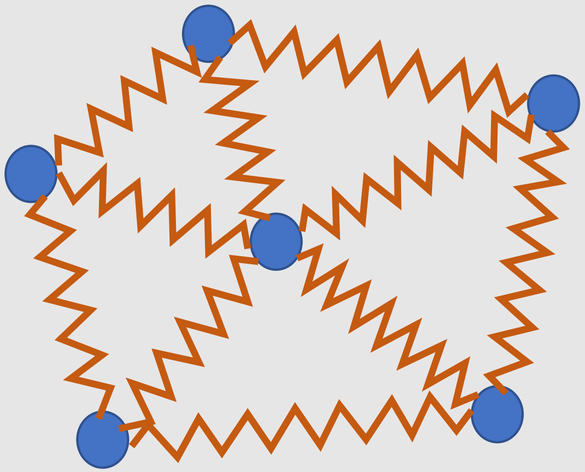


Rigid Body Approximation

剛体近似

Rigid Body Approximation

- In mass-spring system, 5 points in 3D has 15 degrees of freedom

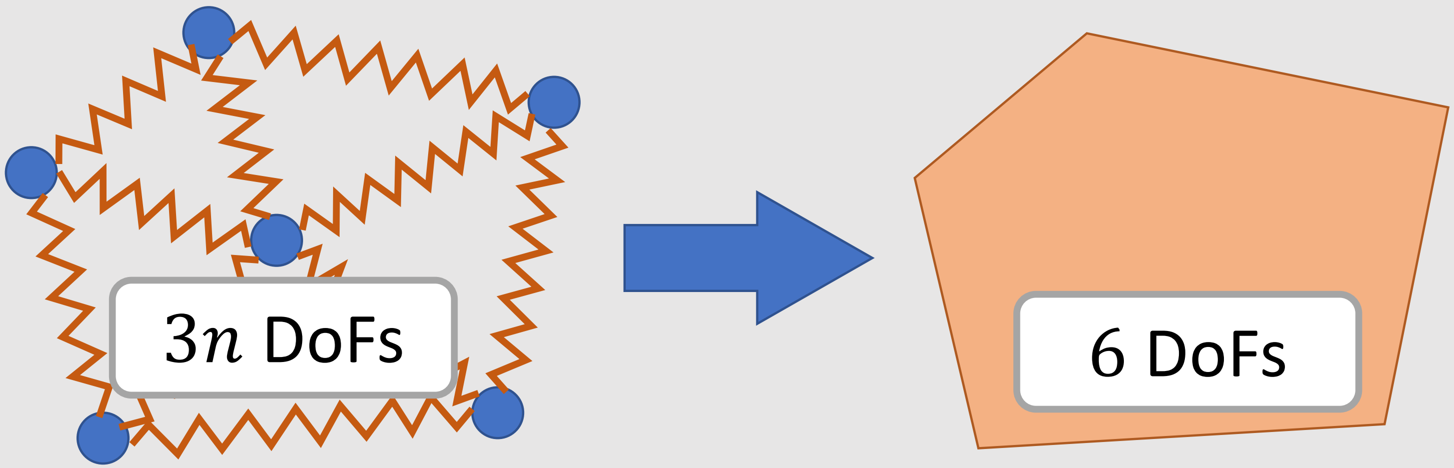


Rigid Body Approximation

- If deformation is negligible, rigid body approximation makes sense

$\vec{x}_{cg}(t)$: the center of gravity's position

$R(t)$: rotation



Rigid Body Approximation

- Equation of motion for rigid body?

$p(t)$: the center of gravity's position

$R(t)$: rotation

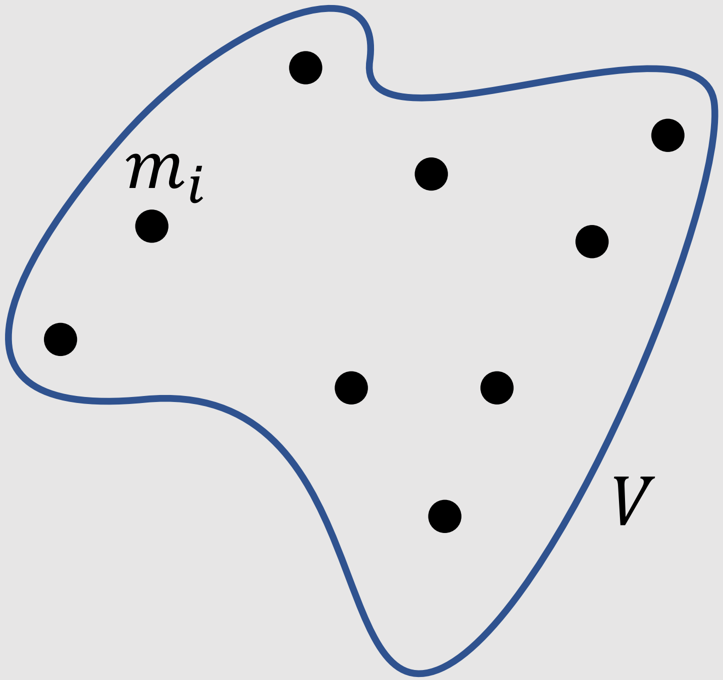
We use Lagrangian Mechanics
to derive equation of motion!

Rigid Body is just an Approximation

- Everything deforms as a reaction to force

Mass(質量)

- Total weight of the object

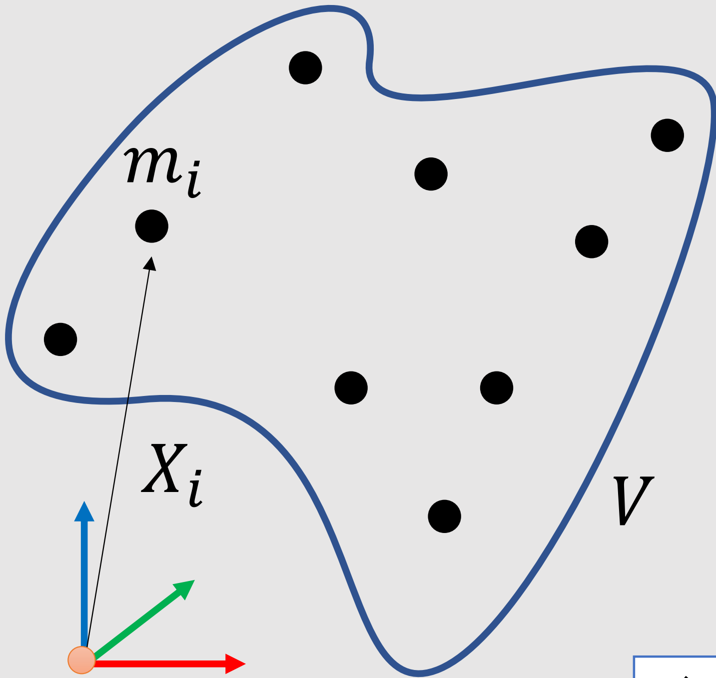


$$M = m_1 + \cdots + m_i$$
$$= \sum_i m_i$$

$$M = \int_V \rho \, dV$$

The Center of the Gravity(重心)

- Average of the positions weighted by mass density



$$\vec{X}_{cg} = \frac{\vec{X}_1 m_1 + \cdots + \vec{X}_i m_i}{m_1 + \cdots + m_i}$$

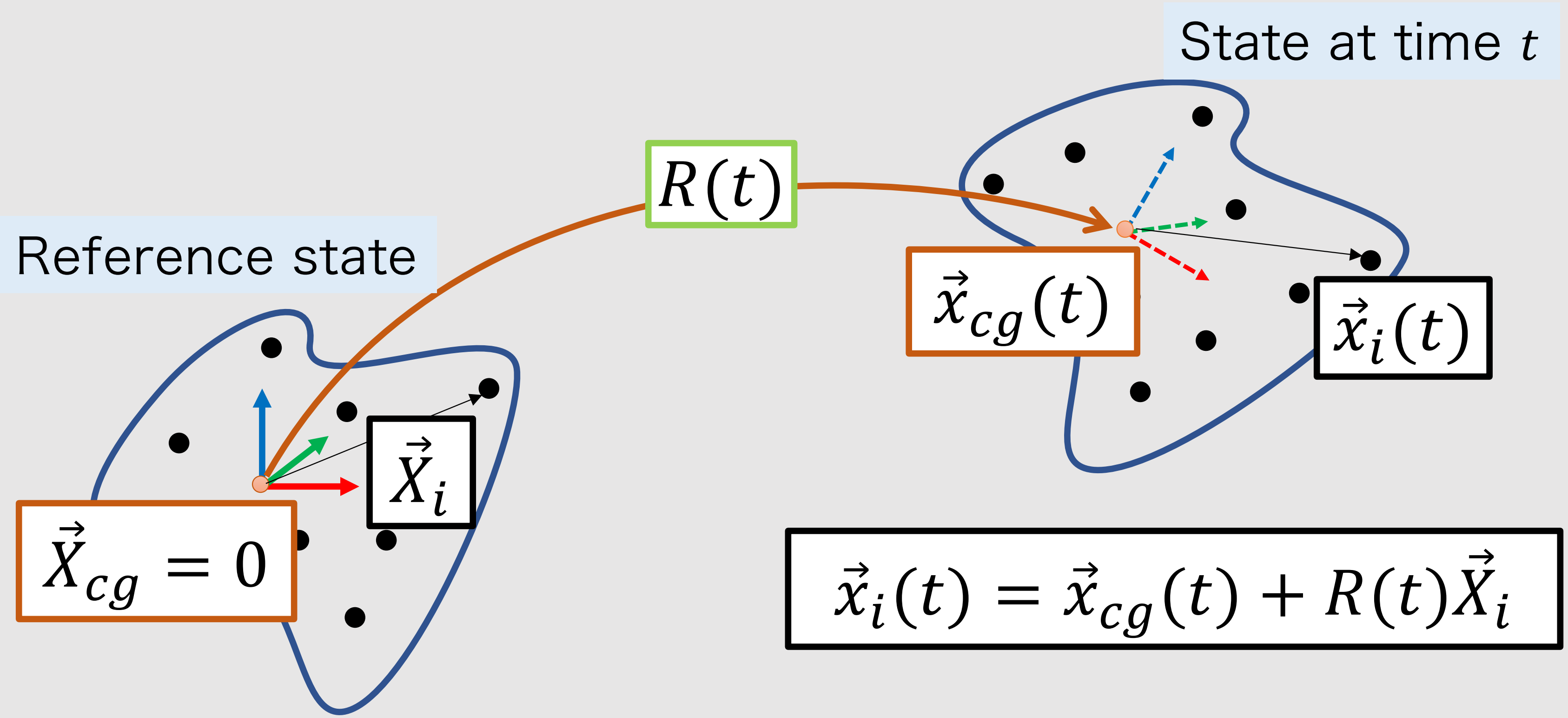
$$= \sum_i \vec{X}_i m_i \Big/ \sum_i m_i$$

$$\vec{X}_{cg} = \int_V \rho \vec{X}\, dV \Big/ \int_V \vec{X}\, dV$$

$\vec{X}_{cg}$ is the centroid(図心) if is constant

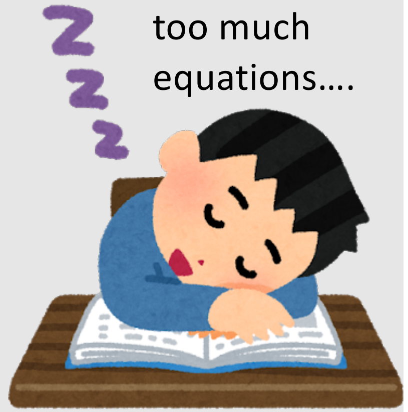
Transformation of a Point on a Rigid Body

- For simplicity, let's put $\vec{X}_{cg}$ at the origin of coordinate: $\vec{X}_{cg} = 0$



$$\vec{x}_i(t) = \vec{x}_{cg}(t) + R(t)\vec{X}_i$$

Linear Momentum (運動量)



$$\vec{x}_i(t) = \vec{x}_{cg}(t) + R(t)\vec{X}_i$$

$$\vec{v}_i(t) = \vec{v}_{cg}(t) + \dot{R}(t)\vec{X}_i$$

$$\dot{R}(t) = R\ \mathrm{Skew}(\vec{\Omega})$$

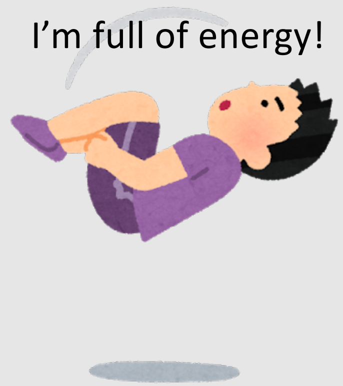
$$\vec{P} = \sum_i m_i \vec{v}_i$$

$$\vec{P} = \sum_i m_i \vec{v}_{cg}(t) + \sum_i m_i R\ \mathrm{Skew}(\vec{\Omega})\vec{X}_i$$

$$\sum_i m_i R\ \mathrm{Skew}(\vec{\Omega})\vec{X}_i = R\ \mathrm{Skew}(\vec{\Omega}) \sum_i m_i \vec{X}_i = R\ \mathrm{Skew}(\vec{\Omega})\ \vec{X}_{cg} = R\ \mathrm{Skew}(\vec{\Omega})\ 0 = 0$$

$$\vec{P} = M\vec{v}_{cg}$$

Kinetic Energy (運動エネルギー)



$$\mathcal{K} = \frac{1}{2}\sum_i m_i \vec{v}_i{}^T \vec{v}_i$$

$$\vec{v}_i(t) = \vec{v}_{cg}(t) + \dot{R}(t)\vec{X}_i$$

$$= \mathrm{R\ Skew}(\vec{\Omega})\vec{X}_i$$
$$= -\mathrm{R\ Skew}(\vec{X}_i)\vec{\Omega}$$

$$\mathcal{K} = \frac{1}{2}\sum_i m_i \vec{v}_{cg}{}^T \vec{v}_{cg} + \frac{1}{2}\sum_i m_i \{R\ \mathrm{Skew}(\vec{X}_i)\vec{\Omega}\}^T \{R\ \mathrm{Skew}(\vec{X}_i)\vec{\Omega}\}$$

inertia tensor

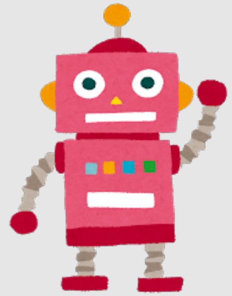
$$\mathcal{K} = \frac{1}{2} M \left\|\vec{v}_{cg}\right\|^2 + \frac{1}{2}\vec{\Omega}(t)^T \left\{-\sum_i m_i \mathrm{Skew}(\vec{X}_i)\ \mathrm{Skew}(\vec{X}_i)\right\}\vec{\Omega}(t)$$

Inertia Tensor(慣性テンソル)

$$I_{in} \equiv -\sum_i m_i \mathrm{Skew}(\vec{X}_i)\,\mathrm{Skew}(\vec{X}_i)$$

$$\Rightarrow \mathcal{K} = \frac{1}{2}M\|\vec{v}_{cg}\|^2 + \frac{1}{2}\vec{\Omega}^T I_{in}\vec{\Omega}$$

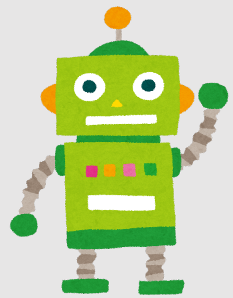
Quadratic form

$\vec{\Omega}$ →  → $\mathcal{K}$

inertia tensor I_{in}
positive semi definite

$$\widetilde{I_{in}} \equiv R I_{in} R^T$$

$$\Rightarrow \mathcal{K} = \frac{1}{2}M\|\vec{v}_{cg}\|^2 + \frac{1}{2}\vec{\omega}^T \widetilde{I_{in}}\vec{\omega}$$

$\vec{\omega}$ →  → $\mathcal{K}$

inertia tensor $\widetilde{I_{in}}$
positive semi definite

Euler’s Rotation Equation

Equation of Motion of Rigid Body

kinetic energy of a point

$$\mathcal{K} = \frac{1}{2}\vec{v}^T m\vec{v}$$

equation of motion

$$m\dot{\vec{v}} = \vec{F}$$

kinetic energy of rigid body

$$\mathcal{K} = \frac{1}{2}\vec{\Omega}^T I_{in}\vec{\Omega}$$

wrong!!

$$\frac{1}{2} I_{in}\dot{\vec{\Omega}} = \vec{F}$$

equation of motion

$$I_{in}\dot{\vec{\Omega}} + \mathrm{Skew}(\vec{\Omega}) I_{in}\vec{\Omega} = \vec{F}$$

Euler-Lagrange Equation

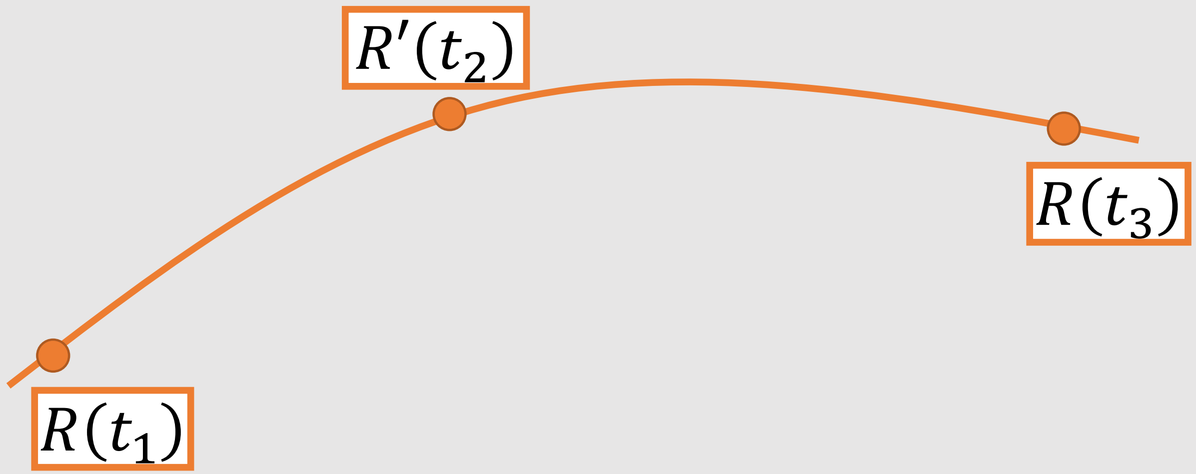
- If $\vec{q}(t)$ is the solution, for arbitrary perturbation $\delta\vec{q}(t)$ it holds:

$$\frac{d}{dt}\left(\frac{\partial \delta \mathcal{L}}{\partial \delta \dot{\vec{q}}}\right) - \frac{\partial \delta \mathcal{L}}{\partial \delta \vec{q}} = 0$$

Parameterization of deviation

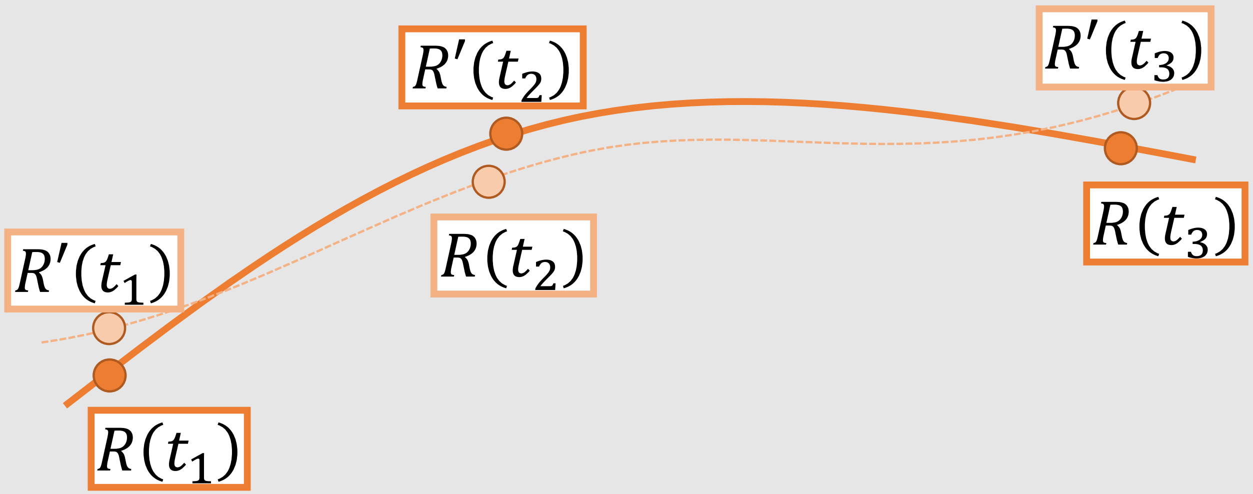
Velocity of the Parameterized deviation
($\vec{\Omega}$ cannot be put in here)

Trajectory of Rotation



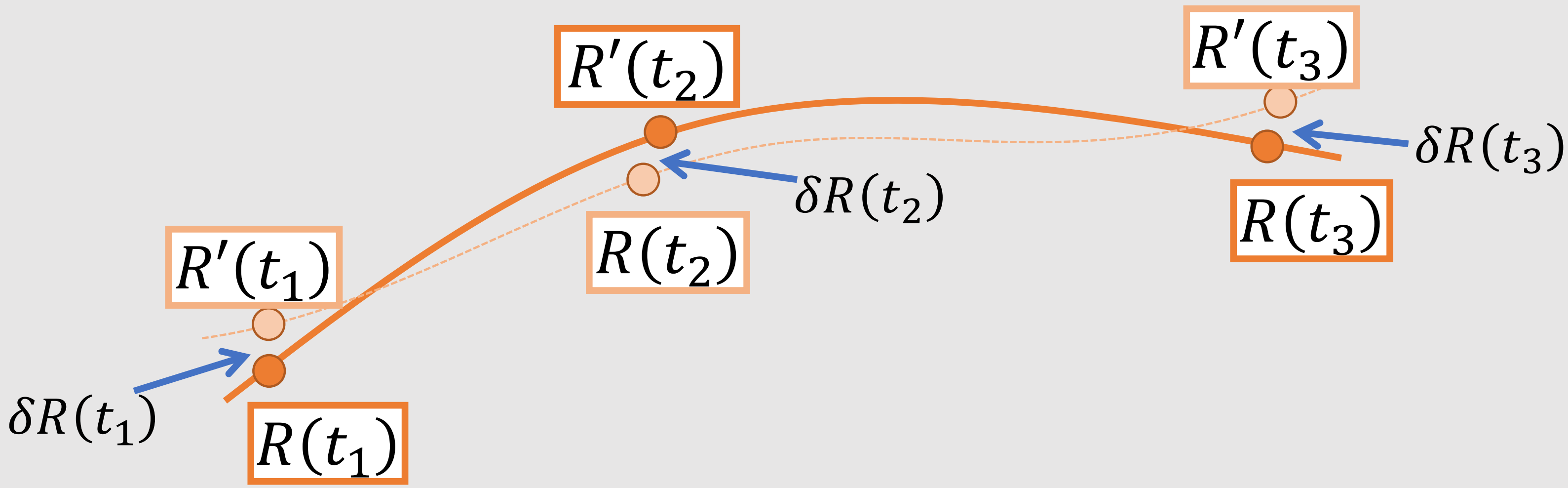
Perturbation of Rotation

Perturbed rotation is constrained as $R'^T R' = I$



Perturbation of Rotation

Perturbation δR get constraint as ${R'}^T R' = (R + \delta R)^T (R + \delta R) = I$



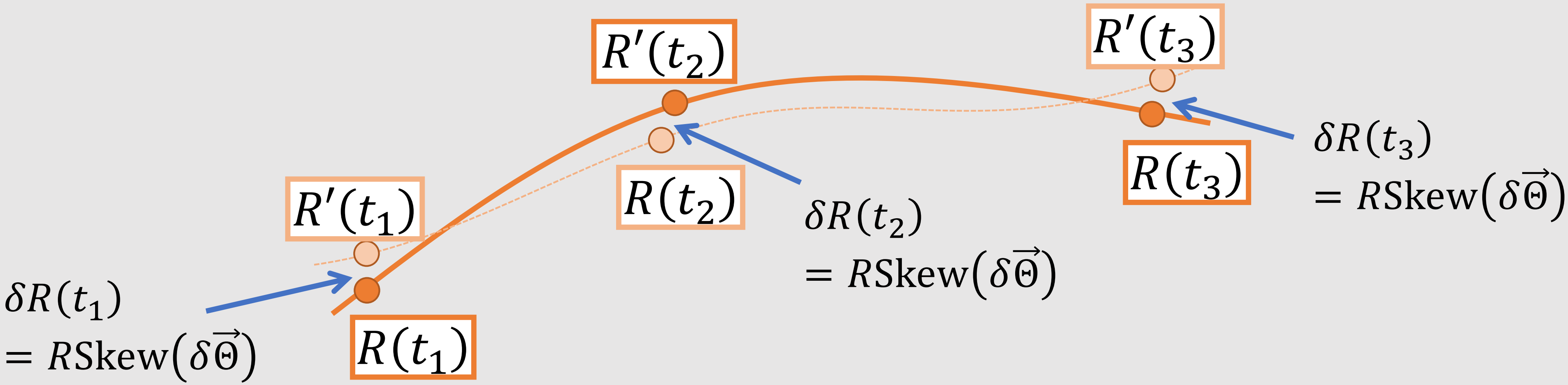
DoF Elimination for Rotation Perturbation

$$R'^{T}R' = (R + \delta R)^{T}(R + \delta R) = I$$

$$\mathrm{Skew}(\delta\vec{\Theta}) \equiv R^{T}\delta R$$

differentiation

$$\mathrm{Skew}\left(\delta\dot{\vec{\Theta}}\right) = \dot{R}^{T}\delta R + R^{T}\delta\dot{R}$$



Perturbation of Angular Velocity

perturbation δR that satisfy constraint

$$\mathrm{Skew}(\delta\vec{\Theta}) \equiv R^T \delta R \qquad \mathrm{Skew}(\vec{\Omega}) = R^T \dot{R}$$

$$\mathrm{Skew}\left(\delta\dot{\vec{\Theta}}\right) = \dot{R}^T \delta R + R^T \delta \dot{R} \qquad \mathrm{Skew}(\delta\vec{\Omega}) = \delta R^T \dot{R} + R^T \delta \dot{R}$$

$$= \mathrm{Skew}(\delta\vec{\Theta})\mathrm{Skew}(\vec{\Omega}) - \mathrm{Skew}(\vec{\Omega})\mathrm{Skew}(\delta\vec{\Theta})$$
$$= \mathrm{Skew}(\mathrm{Skew}(\vec{\Omega})\delta\vec{\Theta})$$

$$\mathrm{Skew}(\delta\vec{\Omega}) = \mathrm{Skew}\left(\delta\dot{\vec{\Theta}}\right) + \delta R^T \dot{R} - \dot{R}^T \delta R$$

$$\delta\vec{\Omega} = \delta\dot{\vec{\Theta}} + \mathrm{Skew}(\vec{\Omega})\delta\vec{\Theta}$$

Rigid Body Floating in Space

- No potential energy & no linear velocity

$$\mathcal{L}(R, \dot{R}) = \mathcal{K} - \mathcal{W} = \frac{1}{2}\vec{\Omega}^T I_{in} \vec{\Omega}$$

$$\delta\mathcal{L}\left(R, \dot{R}, \delta\vec{\Theta}, \delta\dot{\vec{\Theta}}\right) = \delta\vec{\Omega}^T I_{in} \vec{\Omega}$$

$$= \left\{\delta\dot{\vec{\Theta}} + \mathrm{Skew}(\vec{\Omega})\delta\vec{\Theta}\right\}^T I_{in}\,\vec{\Omega}$$

Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial\delta\mathcal{L}}{\partial\delta\dot{\vec{\Theta}}}\right) - \frac{\partial\delta\mathcal{L}}{\partial\delta\vec{\Theta}} = 0$$

equation of motion (a.k.a Euler's equation)

$$\frac{d}{dt}\left(I_{in}\vec{\Omega}\right) + \mathrm{Skew}(\vec{\Omega}) I_{in}\vec{\Omega} = 0$$

Equation of Motion for Rigid Body

equation for reference config

$$I_{in}\dot{\vec{\Omega}} + \mathrm{Skew}\left(\vec{\Omega}\right)I_{in}\vec{\Omega} = 0$$

$$\vec{\omega} = R\vec{\Omega}$$

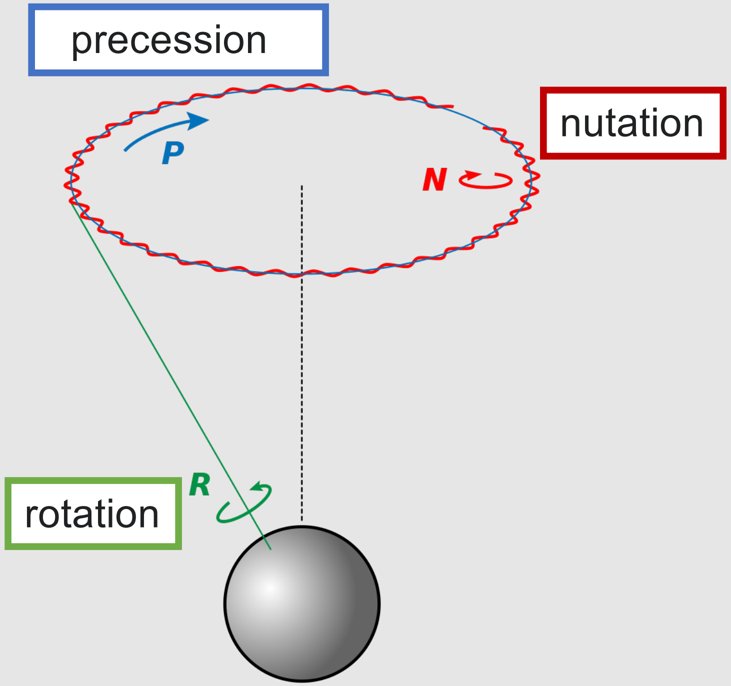
$$\widetilde{I_{in}} = RI_{in}R^T$$

equation for current config

$$\widetilde{I_{in}}\dot{\vec{\omega}} + \mathrm{Skew}(\vec{\omega})\widetilde{I_{in}}\vec{\omega} = 0$$

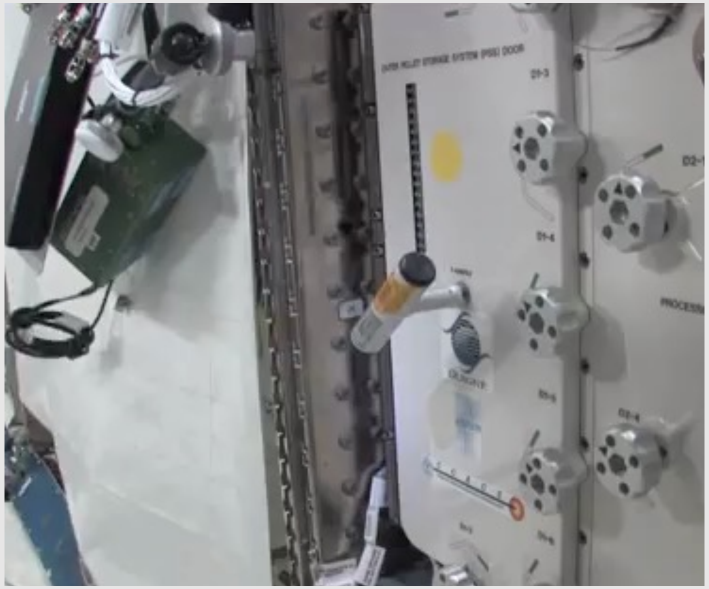
Solution of Euler's Equation

In general, the rotation axis $\vec{\omega}, \vec{\Omega}$ also move around



Credit: User Herbye@Wikipedia

Rotating T-shaped object in zero gravity



Dancing T-handle in zero-g

https://www.youtube.com/watch?v=1n-HMSCDYtM

Solution of Euler's Equation: Special Case

When angular velocity $\vec{\Omega}$ lined up with eigen-vector of inertia tensor I_{in}, angular velocities $\vec{\Omega}$, $\vec{\omega}$ are constant

$$I_{in}\dot{\vec{\Omega}} + \text{Skew}(\vec{\Omega})I_{in}\vec{\Omega} = 0$$

$\lambda\vec{\Omega}$

$\lambda\vec{\Omega}\times\vec{\Omega} = 0$

$$\dot{\vec{\Omega}} = 0$$

$$\vec{\Omega} = constant$$

$$\widetilde{I_{in}}\dot{\vec{\omega}} + \text{Skew}(\vec{\omega})\widetilde{I_{in}}\vec{\omega} = 0$$

$\lambda\vec{\omega}$

$\lambda\vec{\omega}\times\vec{\omega} = 0$

$$\dot{\vec{\omega}} = 0$$

$$\vec{\omega} = constant$$