Rotation

What is Rotation?

- linear transformation R is a rotation if length does not change and volume does not flip

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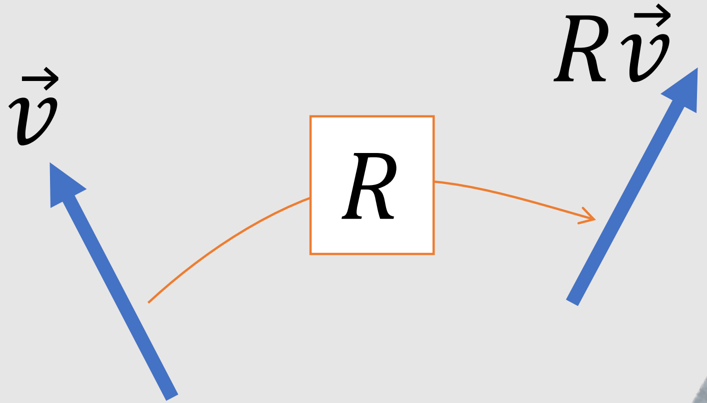


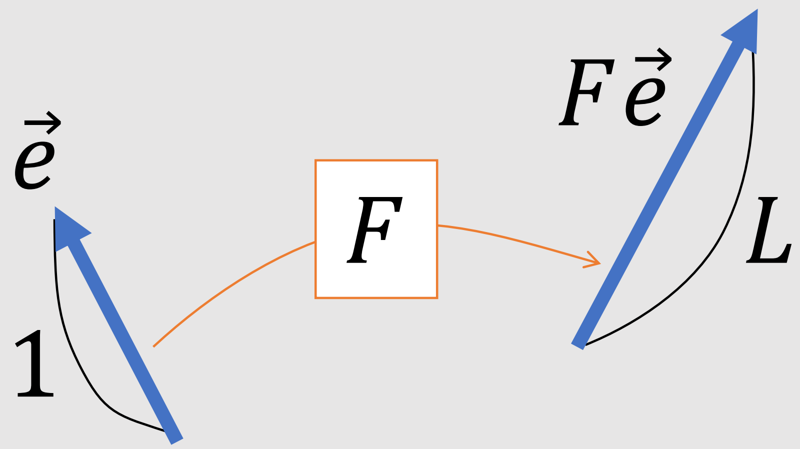
$$\|\vec{v}\|^2 = \|R\vec{v}\|^2$$

$$\vec{v}^T\vec{v} = \vec{v}^T R^T R\vec{v}$$

$\vec{v}$ can be arbitrary!

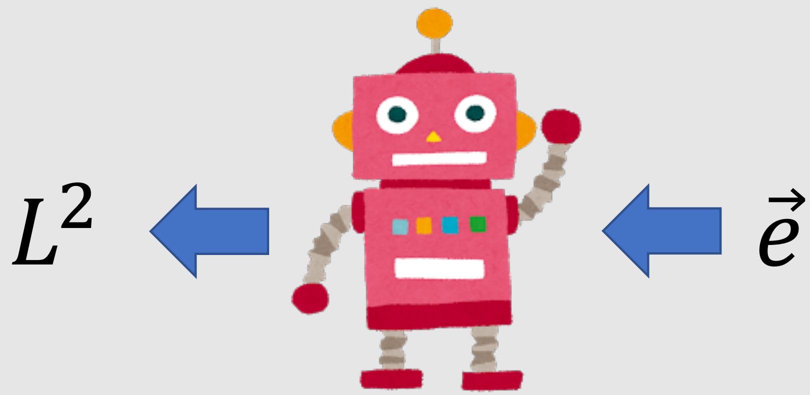
$$R^T R = I$$

Gram Matrix $F^T F$ Stands for Deformed Len.



$$L^2(\vec{e}) = \vec{e}^T F^T F \vec{e}$$

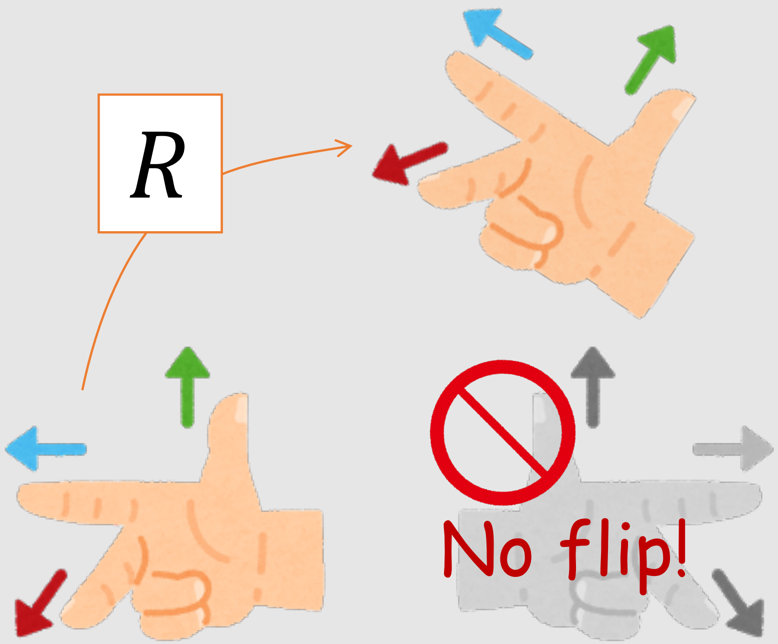
Quadratic form



Positive semi-definite matrix

What is Rotation?

- linear transformation R is a rotation if length does not change and volume does not flip



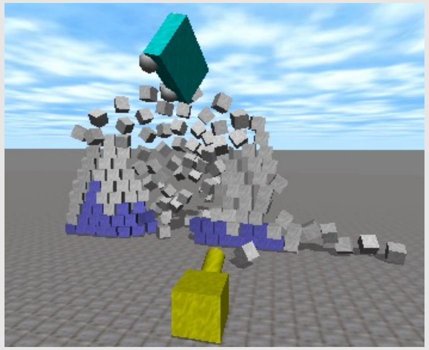
$$\det(R^T R) = \det(\mathrm{I})$$

$$\det(R)^2 = 1$$

$$\det(R) = \pm 1$$

Example of Applications of Rotation

rigid body animation



Credit: Kborer @Wikipedia

character animation

Credit: Banlu Kemiyatorn@Wikipedia

robotics

Credit: Jo Teichmann@Wikipedia

stereo reconstruction

Credit: Kbosak@Wikipedia

Representation of 3D Rotation

For large rotation:

- Rotation matrix
- Axis-angle formulation
- Euler angle
- Quaternion

We four are awesome!

For small rotation:

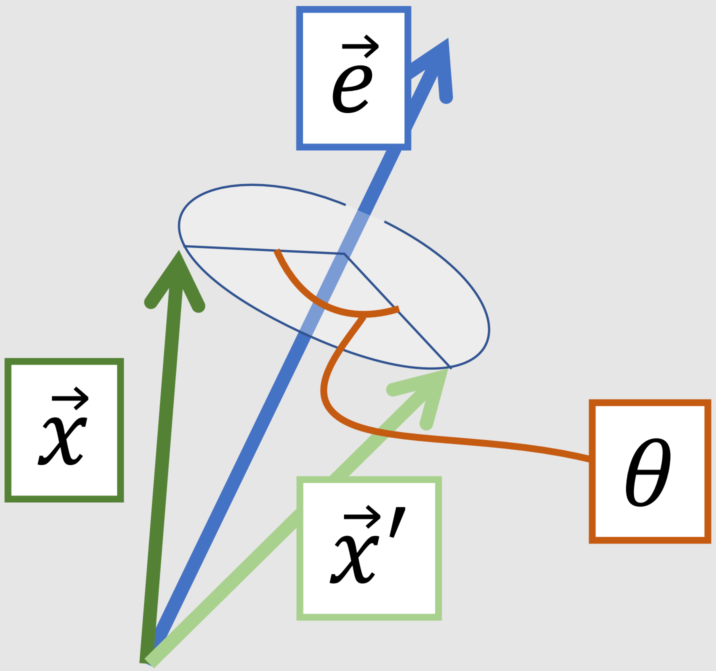
- Infinitesimal rotation

I will be an idol when I grow up!

Rotation Around Axis

- The rotation is parameterized by axis vector $\vec{e}$ and angle θ

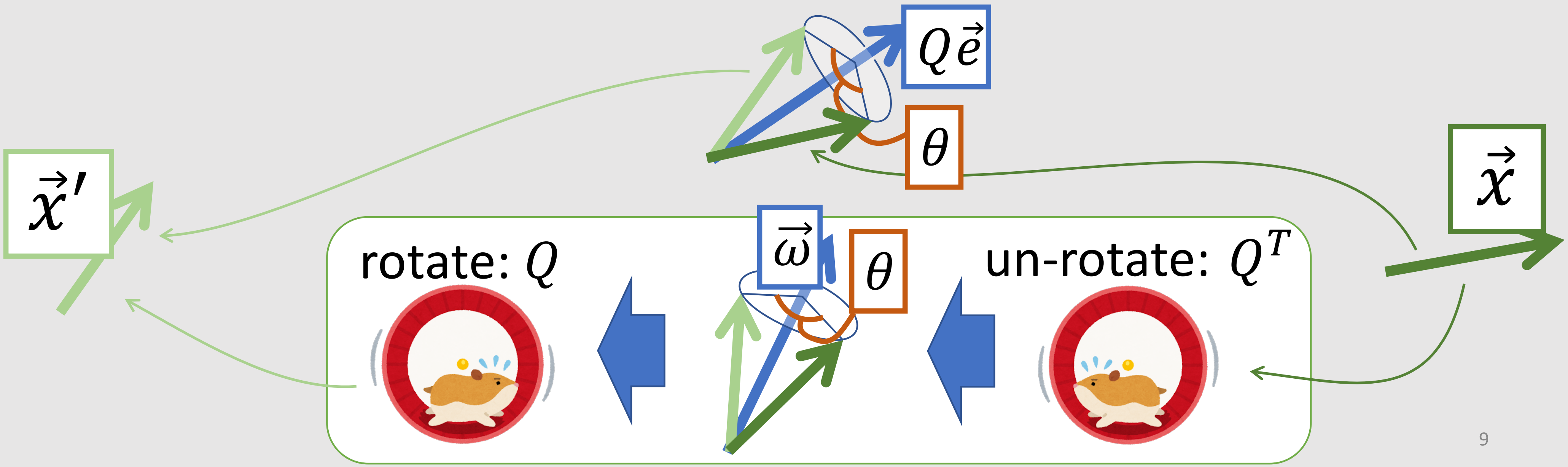
$$\vec{x}' = R(\vec{e}, \theta)\vec{x}$$



Rotation around Rotated Axis

- Axis is rotated with Q → un-rotate the object with Q^T then rotate around $\vec{\omega}$, then rotate back with Q

$$R(Q\vec{\omega}, \theta) = Q * R(\vec{\omega}, \theta) * Q^T$$



intrinsic rotation:
axis rotated

extrinsic rotation:
axis fix

$$R(Q\vec{e},\theta) = Q * \boxed{R(\vec{e},\theta)} * Q^T$$

$$R(Q\vec{e},\theta_2) * Q = Q * R(\vec{e},\theta_2)$$

matrix don't commute!

Rotation around Rotated Axis: Robotic Arm

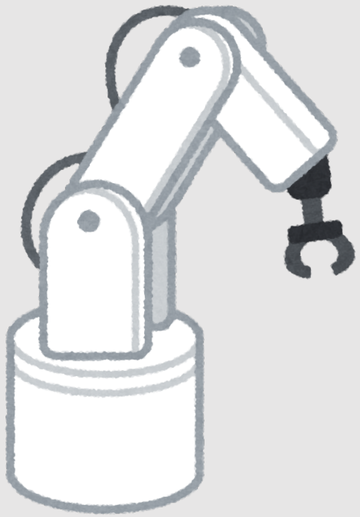
- Rotation of end-effector in a 4-link articulated body

intrinsic

$$R_4(R_3R_2R_1\vec{e}_4, \theta_4) * R_3(R_2R_1\vec{e}_3, \theta_3) * R_2(R_1\vec{e}_2, \theta_2) * R_1(\vec{e}_1, \theta_1)$$

extrinsic

$$R_1(\vec{e}_1, \theta_1) * R_2(\vec{e}_2, \theta_2) * R_3(\vec{e}_3, \theta_3) * R_4(\vec{e}_4, \theta_4)$$



Rotation around Rotated Axis: Gimbal

- Gimbal is used to let object freely rotate (e.g, gyroscope)

Credit: Lucas Vieira @ Wikipedia

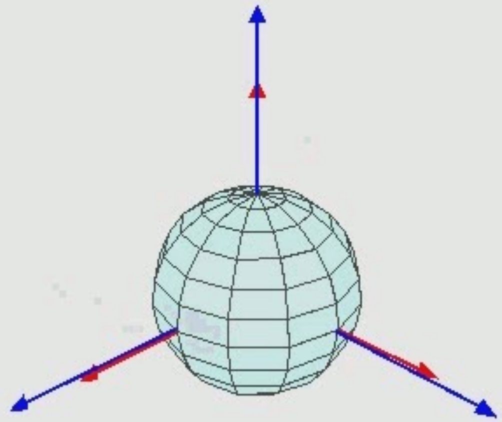
Euler Angle for Rotation Parameterization

intrinsic

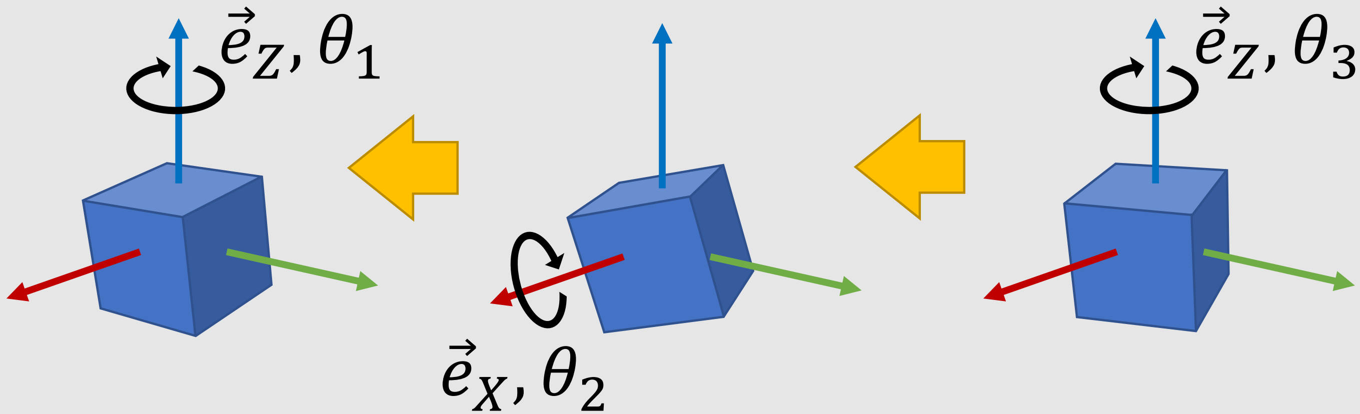
$$\vec{x}' = R_3(R_2 R_1 \vec{e}_Z, \theta_3) * R_2(R_1 \vec{e}_X, \theta_2) * R_1(\vec{e}_Z, \theta_1)\vec{x}$$

extrinsic

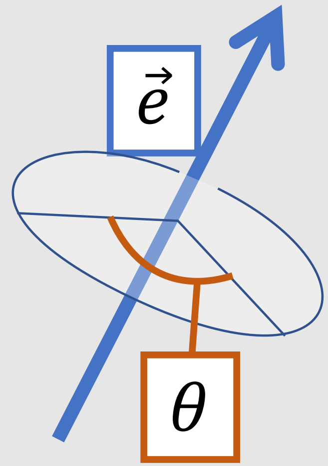
$$\vec{x}' = R_1(\vec{e}_Z, \theta_1) * R_2(\vec{e}_X, \theta_2) * R_3(\vec{e}_Z, \theta_3)\vec{x}$$



credit: Xavax @ Wikipedia



Vector Rotation Parameterizations



Euler vector (rotation vector): $\vec{\omega} = \vec{e}\theta$

Scaling unit axis vector $\vec{e}$ with trigonometric func. of θ

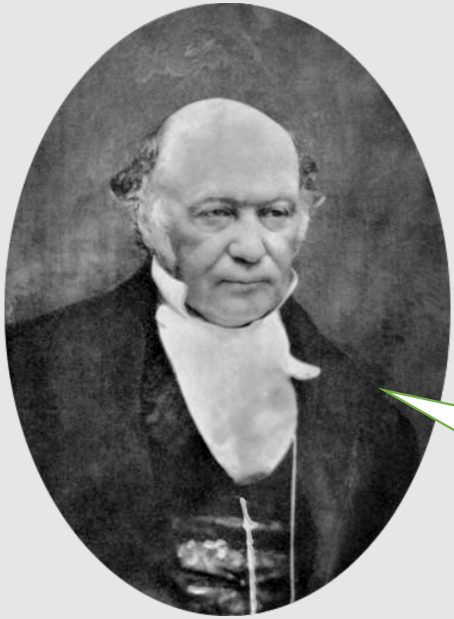
- 360° rotation become zero rotation
- Elegantly composite rotations

Cayley-Gibbs-Rodrigues parameterization: $\vec{e}\tan\frac{\theta}{2}$

Euler-Rodrigues parameterization: $\left(\vec{e}\sin\frac{\theta}{2}, \cos\frac{\theta}{2}\right)$

Discovery of Quaternion

- Sir William Rowan Hamilton (1805–1865) from Ireland

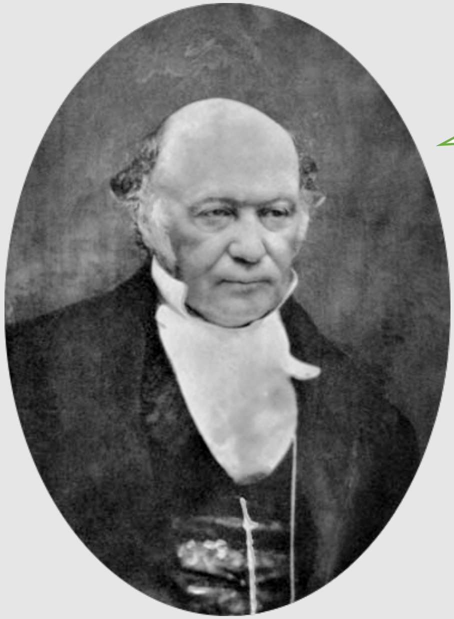


Pa, imaginary number for 3D yet?

No clue for 10 years... 😂
Let me take a walk for a change...

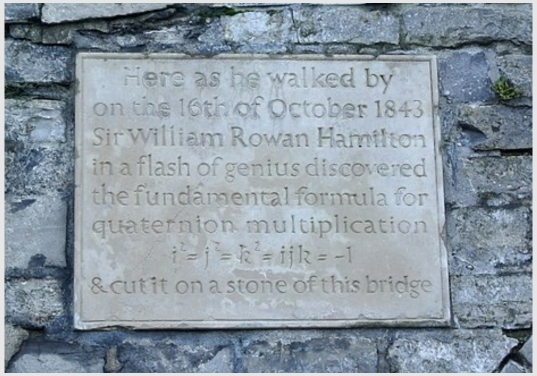
Discovery of Quaternion

- Sir William Rowan Hamilton (1805–1865) from Ireland



Finally got an idea 🎉
$$i^2 = j^2 = k^2 = ijk = -1$$

Credit: Wisher@wikipedia



Credit: Cone83@wikipedia

Properties of Quaternion

- One real number and three imaginary numbers

$$q = r + v_x\boldsymbol{i} + v_y\boldsymbol{j} + v_z\boldsymbol{k}$$

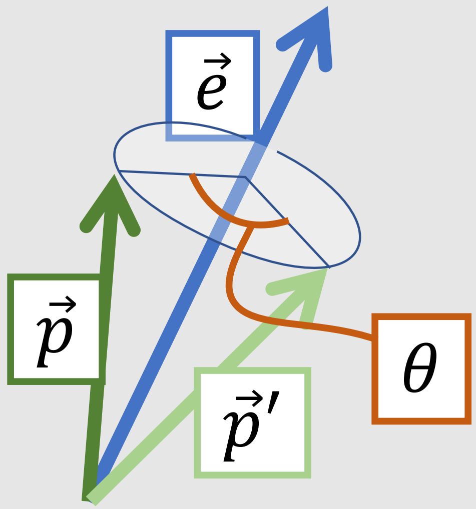
- Conjugate quaternion

$$\bar{q} = r - v_x\boldsymbol{i} - v_y\boldsymbol{j} - v_z\boldsymbol{k}$$

- Norm of quaternion

$$\|q\|^2 = q\bar{q} = r^2 + {v_x}^2 + {v_u}^2 + {v_z}^2$$

Rotation with Quaternion



Euler-Rodrigues parameterization

$$q = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\left(e_x\boldsymbol{i} + e_y\boldsymbol{j} + e_z\boldsymbol{k}\right)$$

$$p = p_x\boldsymbol{i} + p_y\boldsymbol{j} + p_z\boldsymbol{k}$$

$$p' = p'_x\boldsymbol{i} + p'_y\boldsymbol{j} + p'_z\boldsymbol{k}$$

input vector

$$p' = qp\bar{q}$$

rotated vector

Composite Rotation

- Multiplication of quaternion is associative

rotation with q_1 and then q_2

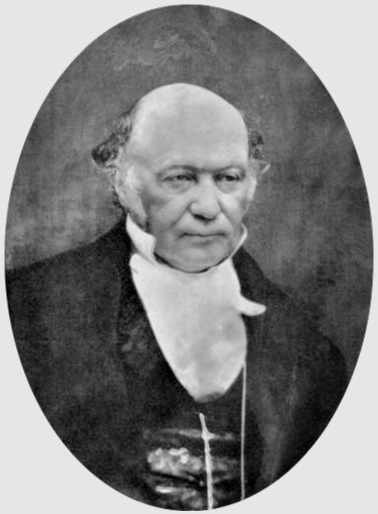
$$q_{12} = q_2 q_1$$

quaternion don't commute!

Cayley–Hamilton Theorem

- Eigen values are the root of the characteristic polynomial

$$p(\lambda) = \det(A - \lambda I) = \sum c_i \lambda^i = 0$$



Characteristic polynomial of a matrix A produces zero , when inputting A

$$p(A) = \sum c_i A^i = 0$$

Cayley–Hamilton Theorem for a 3x3 Matrix

- Characteristic polynomial of a matrix A produces zero , when inputting A

$$p(A) = A^3 + c_1 A^2 + c_2 A + c_3 I = 0$$

$$c_1 = \mathrm{tr(A)} = \lambda_1 + \lambda_2 + \lambda_3$$
$$c_2 = 1/2\,\{\mathrm{tr}(A)^2 - tr(A^2)\} = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1$$
$$c_3 = \det(A) = \lambda_1\,\lambda_2\,\lambda_3$$

Infinitesimal Rotation is a Vector

- What if the rotation is very small?

$$R = I + \Omega$$

$$R^T R = (I + \Omega)^T (I + \Omega) = I$$

$$\Omega^T = -\Omega$$

Ω is a skew symmetric matrix!

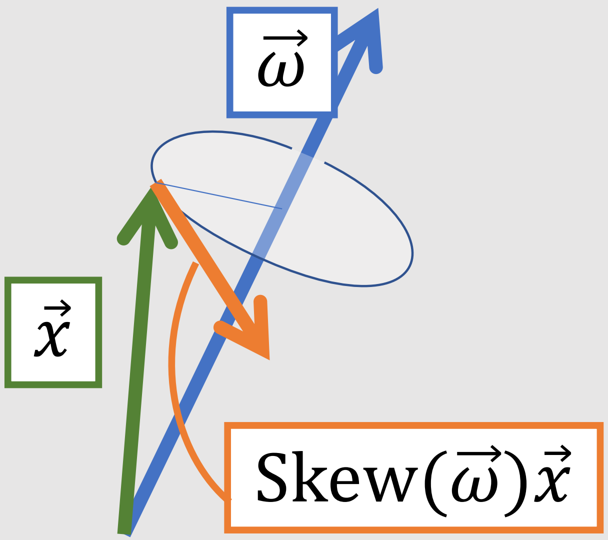
Properties of 3x3 Skew Symmetric Matrix

- Skew matrix has only three independent elements

$$\Omega = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \equiv \mathrm{Skew}(\vec{\omega})$$

- Skew matrix defines cross product

$$\Omega\vec{v} = \vec{\omega}\times\vec{v} \qquad \vec{\omega} = (\omega_1, \omega_2, \omega_3)^T$$



3x3 Skew Symmetric Matrix **Squared**

$$\Omega^2 = \text{Skew}(\vec{\omega})^2 = \begin{bmatrix} -\omega_2{}^2 - \omega_3{}^2 & \omega_1\omega_2 & \omega_1\omega_3 \\ \omega_1\omega_2 & -\omega_3{}^2 - \omega_1{}^2 & \omega_2\omega_3 \\ \omega_1\omega_3 & \omega_2\omega_3 & -\omega_1{}^2 - \omega_2{}^2 \end{bmatrix}$$

$$= \vec{\omega} \otimes \vec{\omega} - \vec{\omega}^T \vec{\omega} I$$

$$\text{tr}(\Omega^2) = \omega_1{}^2 + \omega_2{}^2 + \omega_3{}^2 - 3(\omega_1{}^2 + \omega_2{}^2 + \omega_3{}^2)$$

$$= -2\|\vec{\omega}\|^2$$

3x3 Skew Symmetric Matrix Cubed

Cayley-Hamilton's theorem

$$p(\Omega) = \Omega^3 + (\mathrm{tr}\Omega)\Omega^2 + \frac{(\mathrm{tr}\Omega)^2 - \mathrm{tr}(\Omega^2)}{2}\Omega + (\det\Omega)I = 0$$

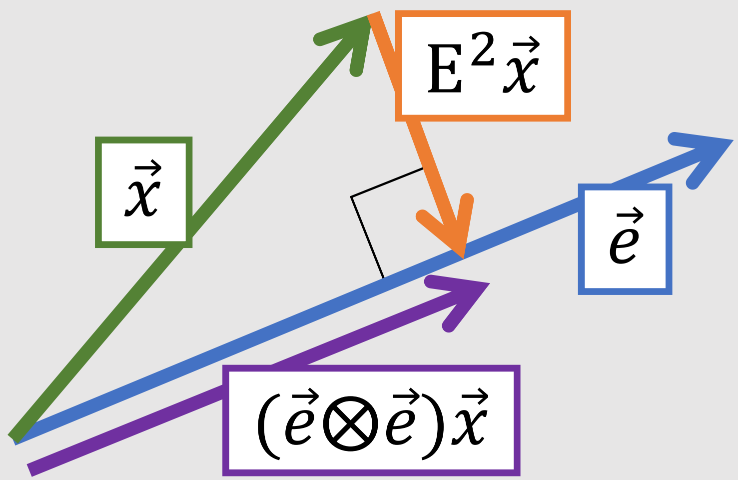
$$\mathrm{tr}\,\Omega = 0,\ \det\Omega = 0, \mathrm{tr}(\Omega^2) = -2\|\vec{\omega}\|^2$$

$$\Omega^3 = -\|\vec{\omega}\|^2\Omega$$

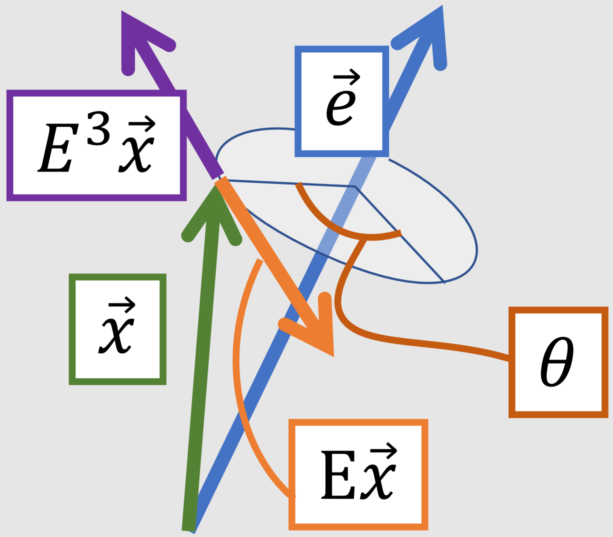
Skew Symmetric Matrix of Unit Vector $\vec{e}$

$$E = \text{Skew}(\vec{e}), \qquad where\ \|\vec{e}\| = 1$$

$$E^2 = \vec{e} \otimes \vec{e} - I$$



$$E^3 = -E$$



From Infinitesimal Rotation Vector to Matrix

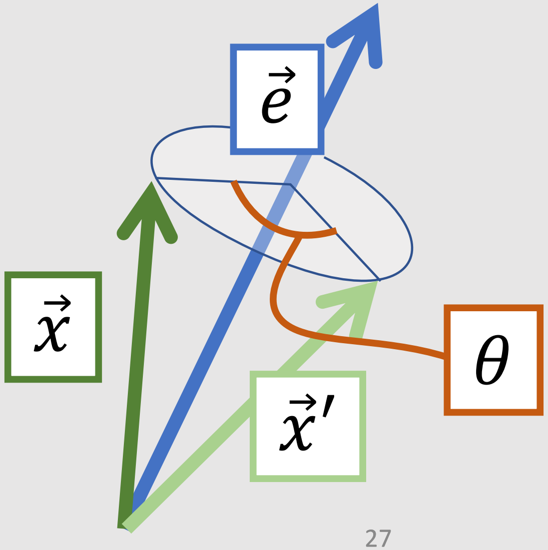
$$\mathrm{E} = \mathrm{Skew}(\vec{e}) = \begin{bmatrix} 0 & e_3 & -e_2 \\ -e_3 & 0 & e_1 \\ e_2 & -e_1 & 0 \end{bmatrix}$$

$$R(\vec{e},\theta) = \exp(\theta E)$$

$$= \mathrm{I} + \frac{\theta}{1!}E + \frac{\theta^2}{2!}E^2 + \frac{\theta^3}{3!}E^3 + \frac{\theta^4}{4!}E^4 \ldots$$

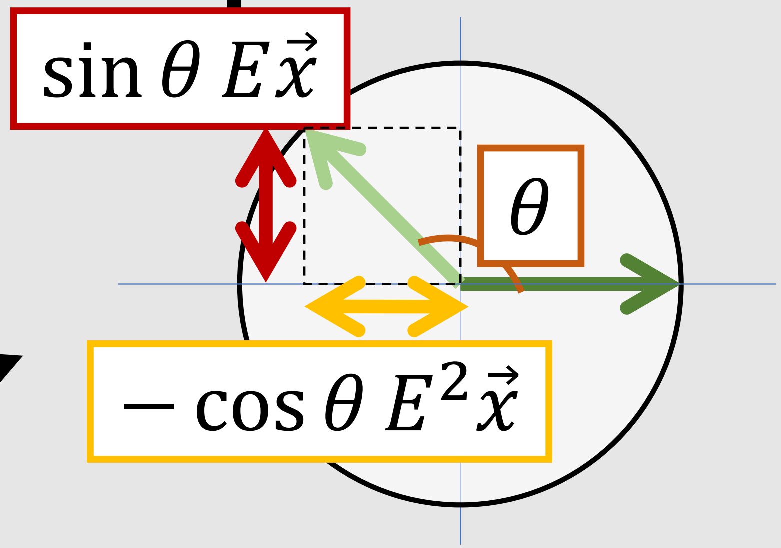
$$= \mathrm{I} + \sin\theta E + (1-\cos\theta)E^2$$

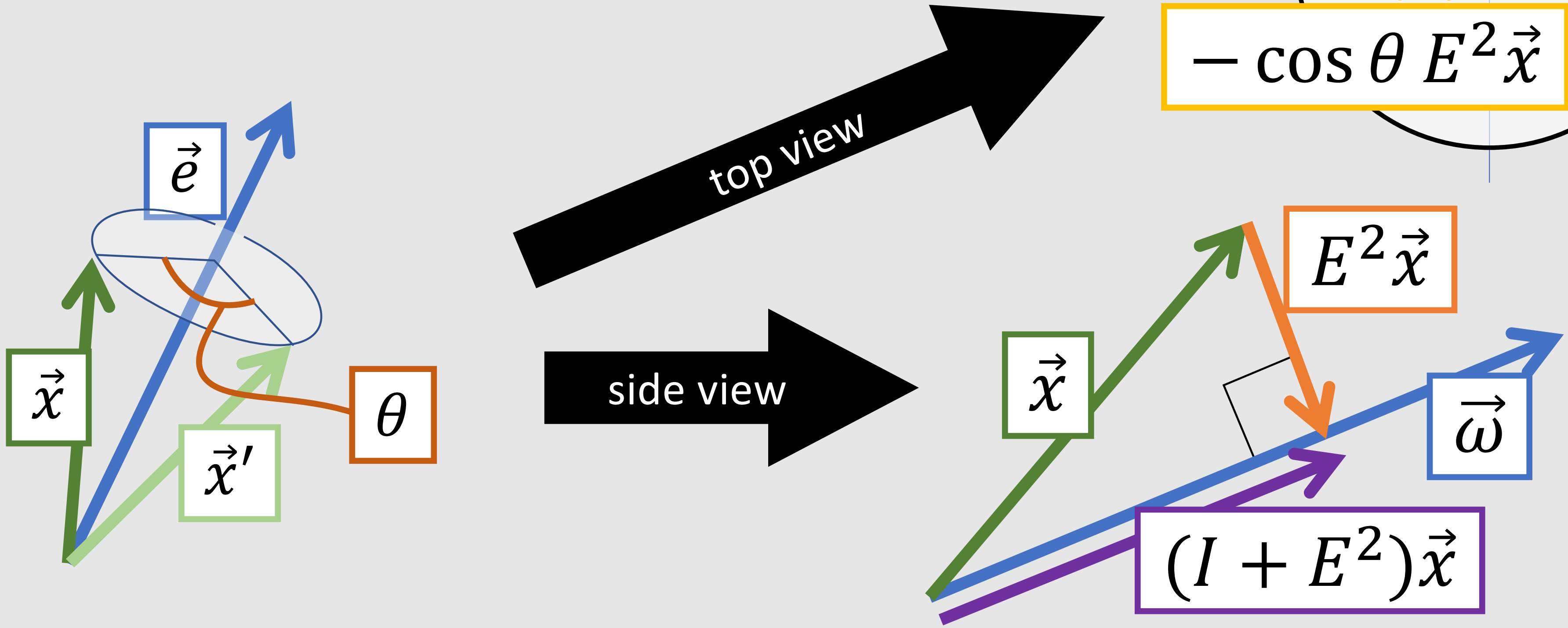
Rodriguez's rotation formula



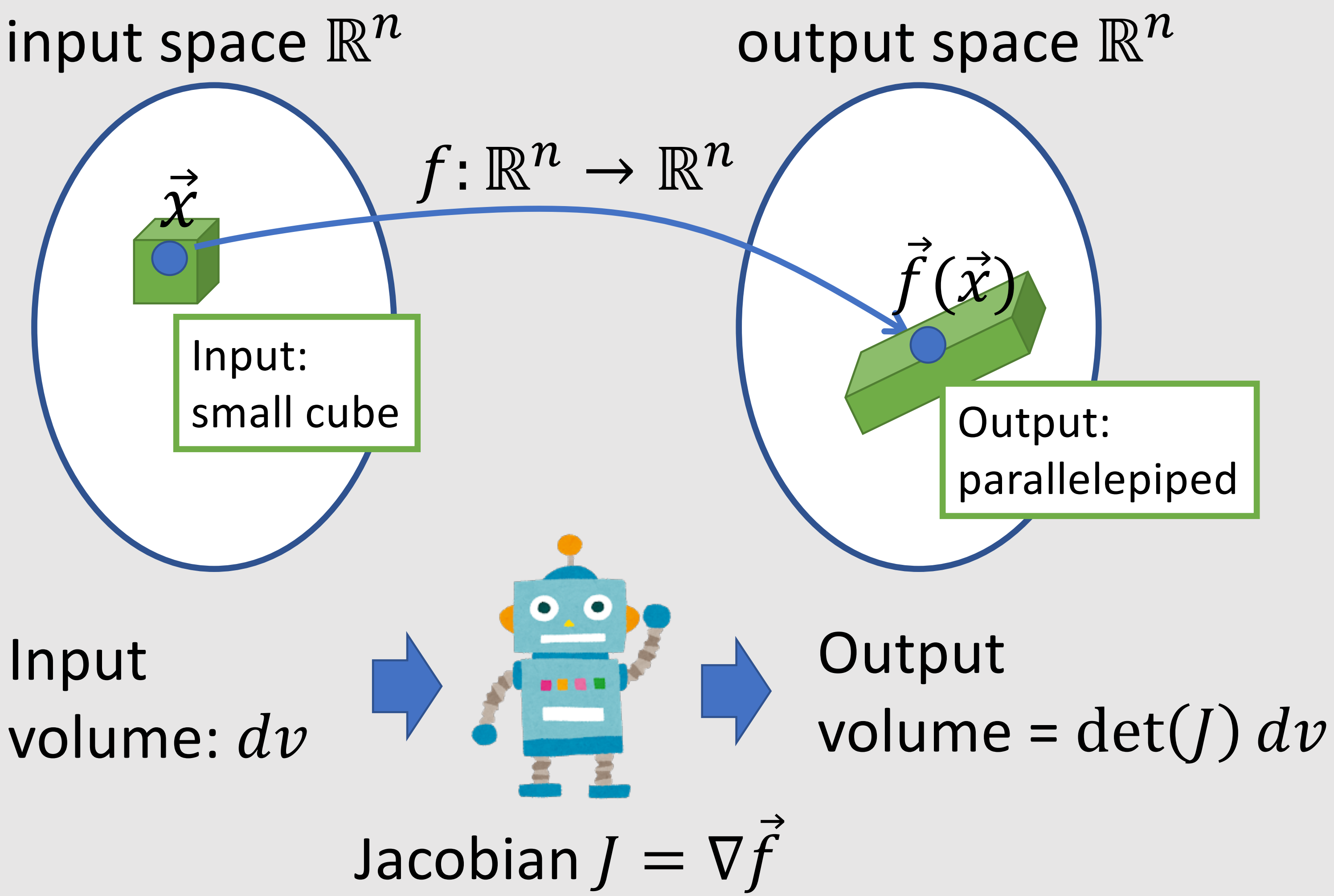
Rodriguez's Rotation Formula Explained

$$R(\vec{e},\theta) = \mathrm{I} + \ \sin\theta E + (1 - \cos\theta)E^2$$





Jacobian Determinant: Volume Change Ratio



Jacobian Determinant: Volume Change Ratio

input:

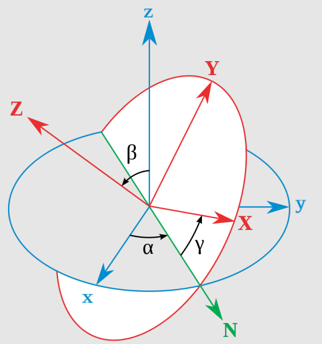
small change in Euler angle

$(d\theta_1, d\theta_2, d\theta_3)$

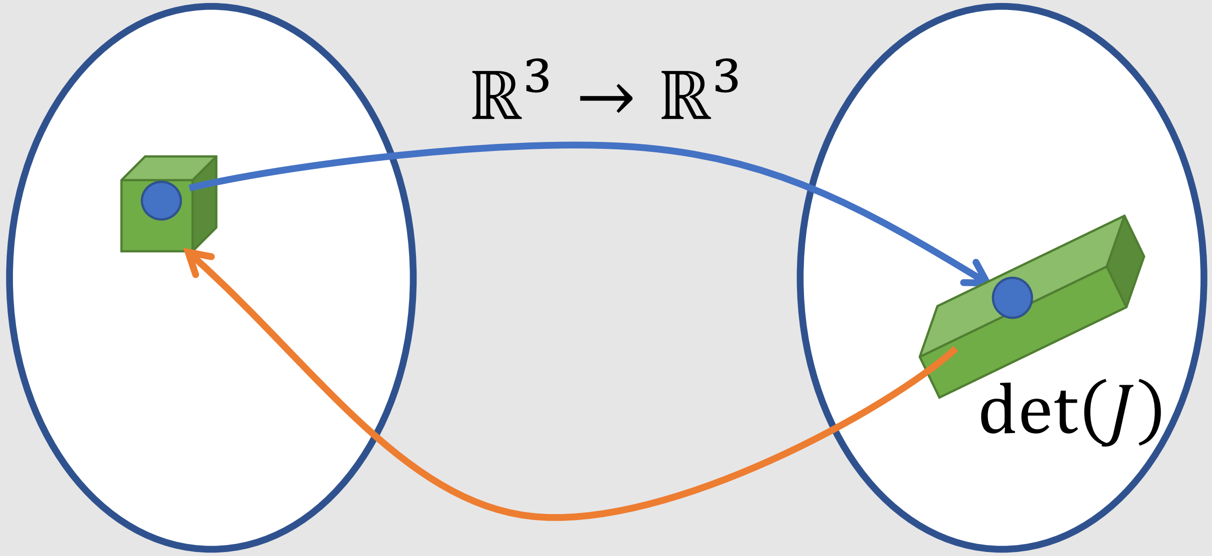
output:

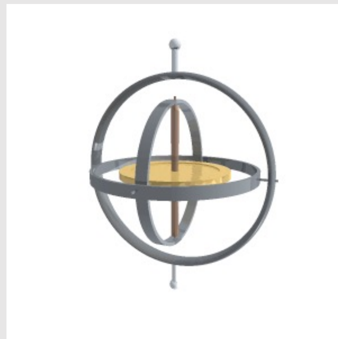
infinitesimal rotation

$d\vec{\omega} = (d\omega_1, d\omega_2, d\omega_3)$



Credit: Lionel Brits @ Wikipedia





Credit: Lucas Vieira @Wkipedia

If output volume is not zero $\det(J) \neq 0$, there is an inverse map

Gimbal Lock: Zero Jacobian Determinant

input:

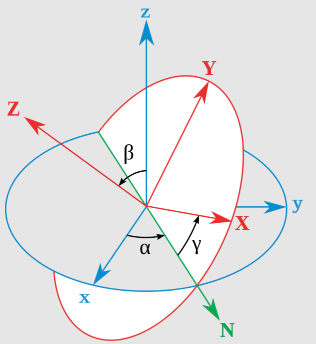
small change in Euler angle

$(d\theta_1, d\theta_2, d\theta_3)$

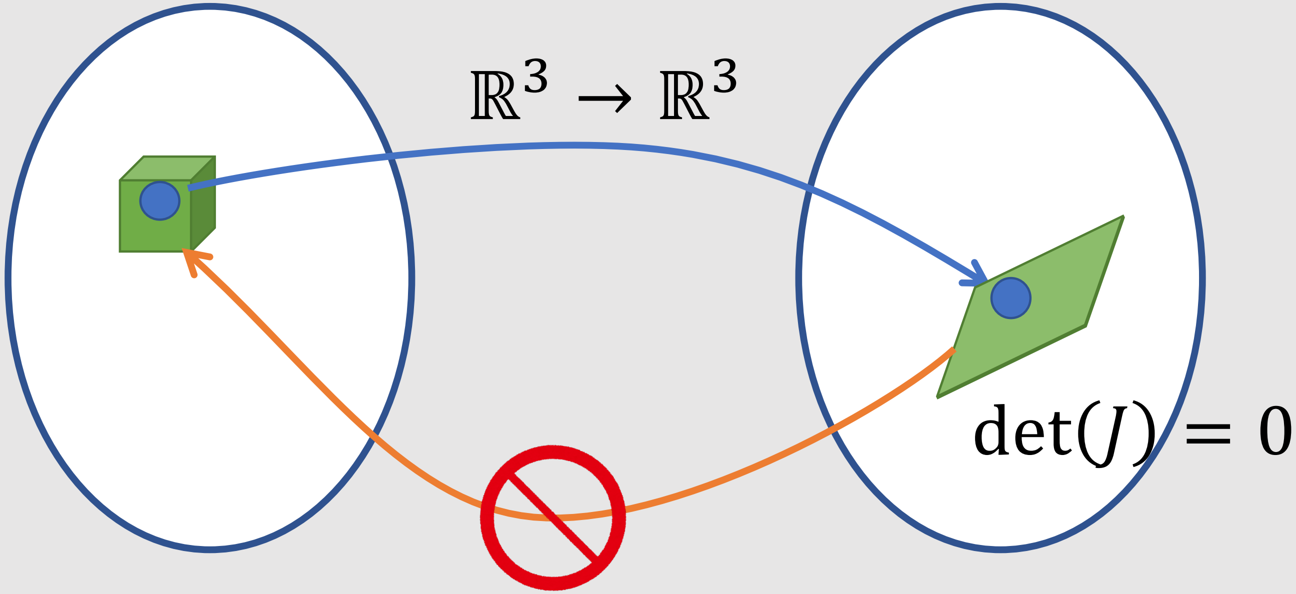
output:

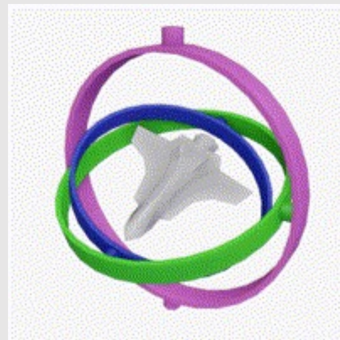
infinitesimal rotation

$\mathrm{d}\vec{\omega} = (d\omega_1, d\omega_2, d\omega_3)$



Credit: Lionel Brits @ Wikipedia

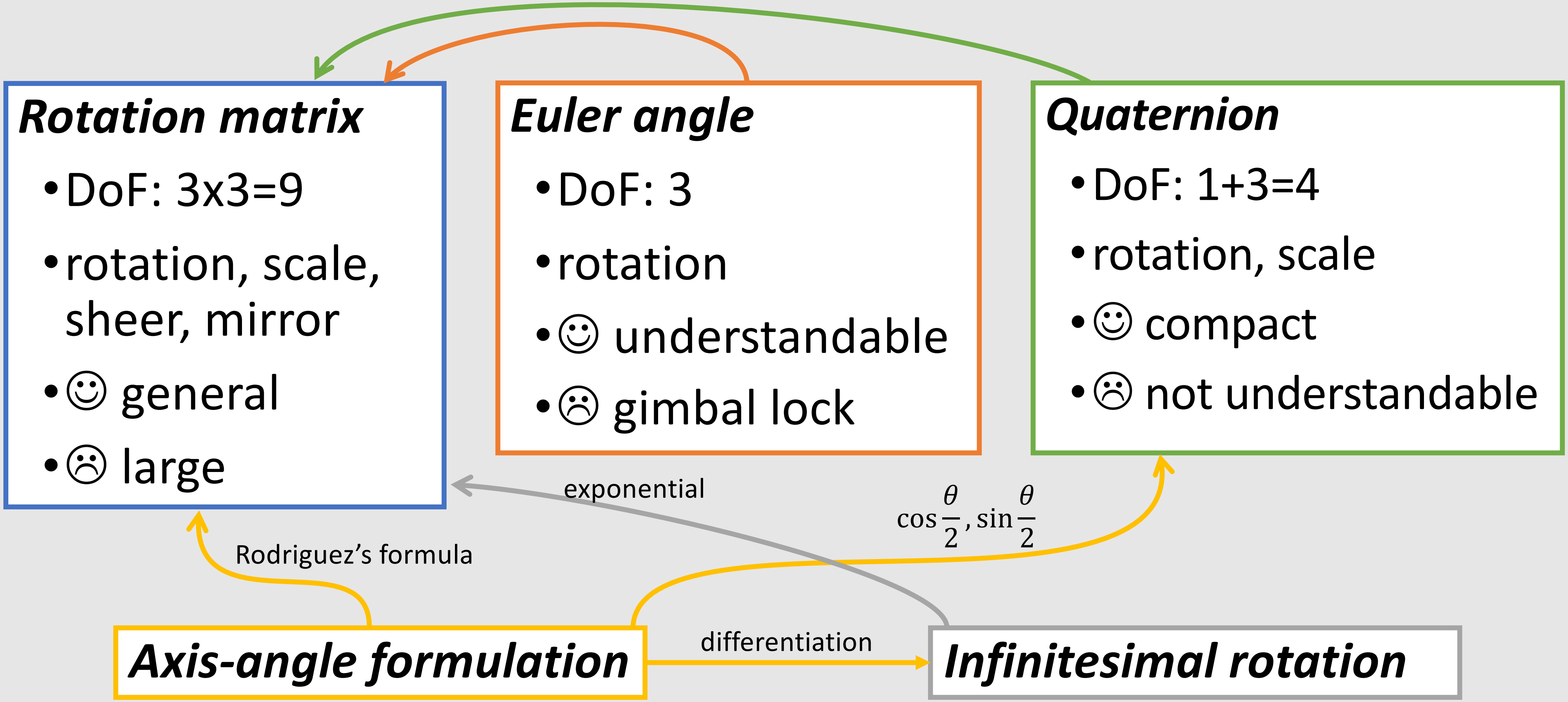




Credit: Drummyfish @Wikipedia

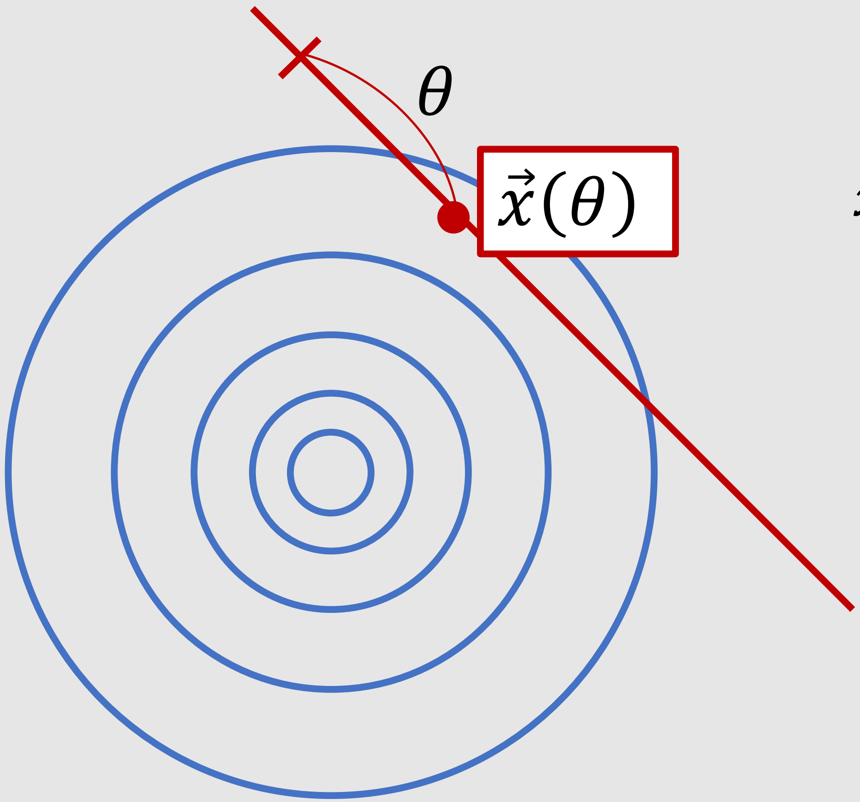
no inverse map when axes aligned!

Comparison of 3D Rotation Representations



DoF Elimination using Parameterization

Parameterize solution $\vec{x}(\theta)$ such that constraints naturally satisfy



$$\underset{\vec{x}\subset\{\vec{x}|g(\vec{x})=0\}}{\operatorname{argmin}} W(\vec{x}) \Rightarrow \underset{\theta}{\operatorname{argmin}} W(\vec{x}(\theta))$$

e.g., $g(\vec{x}) = x + y + 2 = 0$

$$\Rightarrow \begin{cases} x = +\theta - 1 \\ y = -\theta - 1 \end{cases}$$

Minimize Parameterized Solution

$$\underset{\theta}{\operatorname{argmin}} W(\vec{x}(\theta))$$

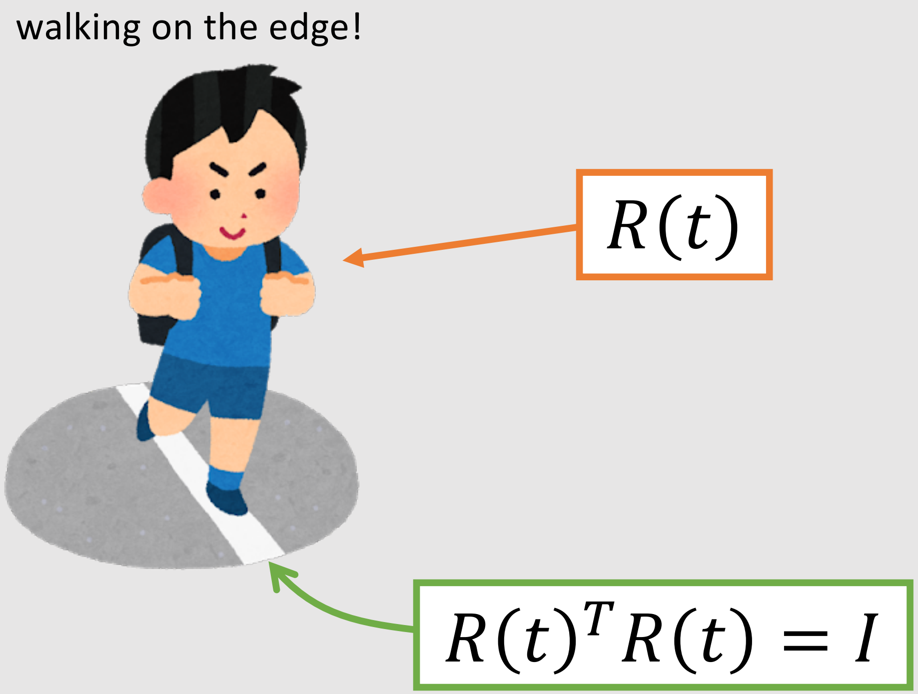
↓ Newton-Raphson method

$$d\theta = -\left[\frac{\partial^2 W}{\partial \theta^2}\right]^{-1} \left(\frac{\partial W}{\partial \theta}\right)$$

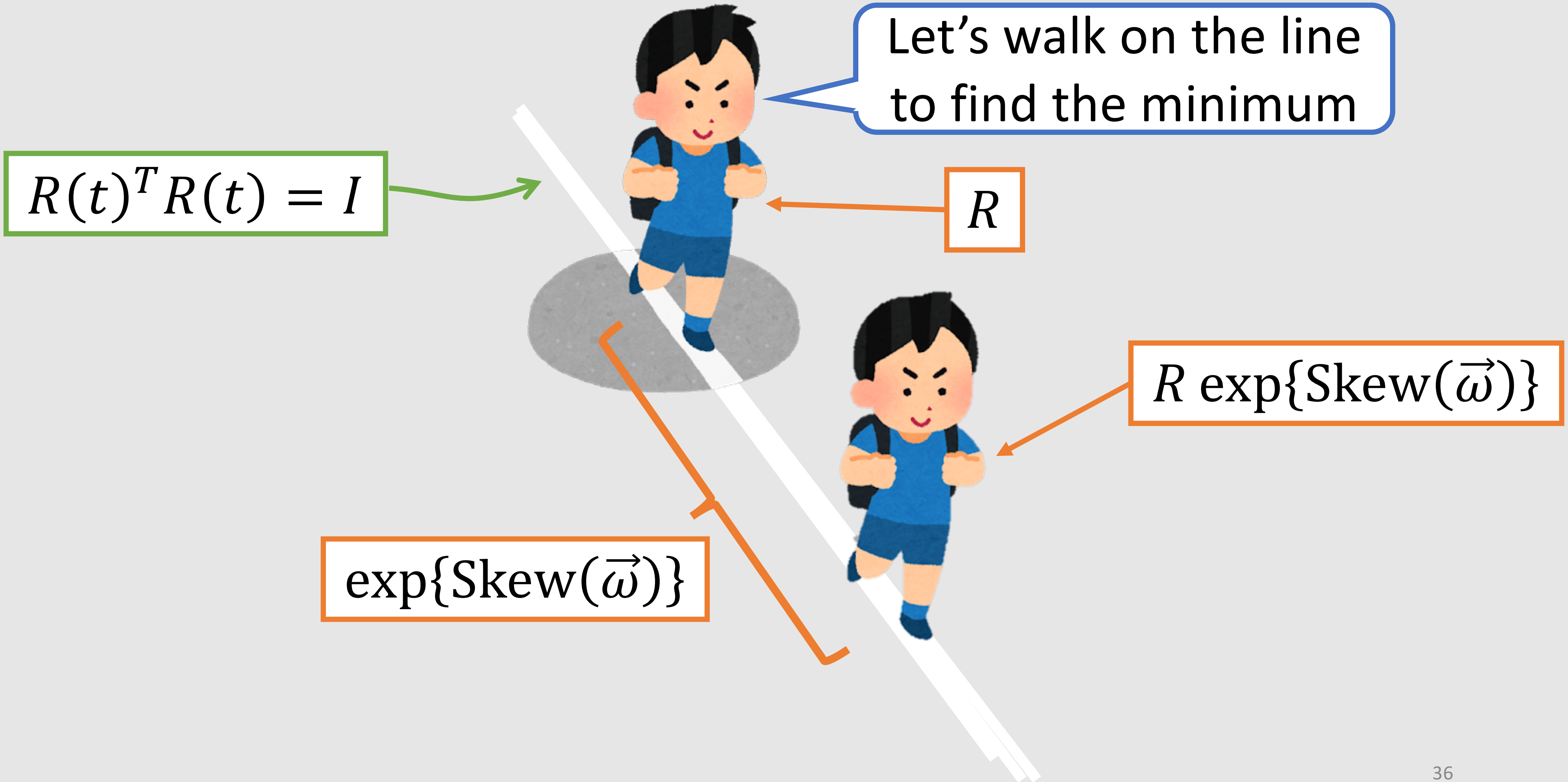
find the root of gradient!

Differentiation of Rotation Matrix

Change under constraints $R(t)^T R(t) = I$ → DoF elimination



Parameterization of Rotation



Optimization of Rotation: Pattern R*dR

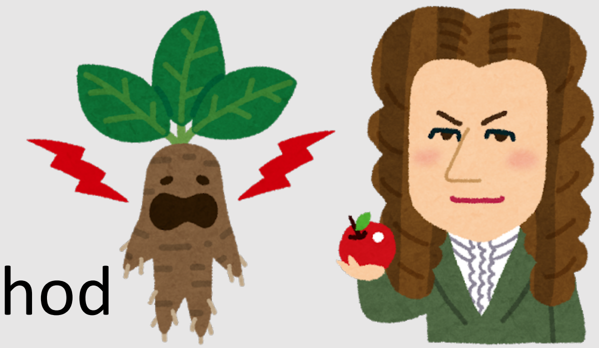
$$R = R_i \exp\{\mathrm{Skew}(\vec{\omega})\}$$

Minimize $W(R)$

Minimize $W(R(\vec{\omega}))$

Newton method

find the root of gradient!



$$\vec{\omega} = -\left[\frac{\partial^2 W}{\partial^2 \vec{\omega}}\right]^{-1} \left(\frac{\partial W}{\partial \vec{\omega}}\right)$$

$$R_{i+1} = R_i \exp\{(\mathrm{Skew}(\vec{\omega})\}$$

$$\exp\{\mathrm{Skew}(\vec{\omega})\} = I + \mathrm{Skew}(\vec{\omega}) + \frac{1}{2}\mathrm{Skew}(\vec{\omega})^2 \cdots$$

Optimization of Rotation: Pattern dR*R

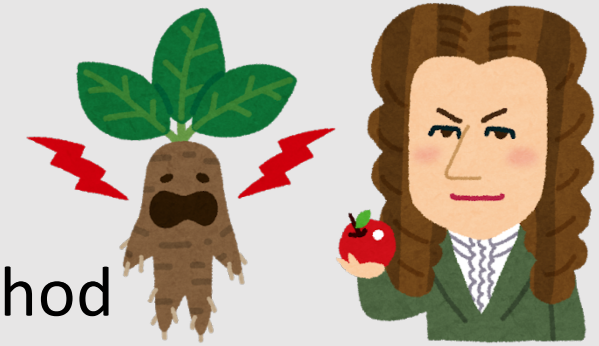
$$R = \exp\{\mathrm{Skew}(\vec{\omega})\}\, R_i$$

Minimize $W(R)$

Minimize $W(R(\vec{\omega}))$

Newton method

find the root of gradient!



$$\vec{\omega} = -\left[\frac{\partial^2 W}{\partial^2 \vec{\omega}}\right]^{-1} \left(\frac{\partial W}{\partial \vec{\omega}}\right)$$

$$R_{i+1} = \exp\{(\mathrm{Skew}(\vec{\omega})\}\, R_i$$

$$\exp\{\mathrm{Skew}(\vec{\omega})\}$$
$$= I + \mathrm{Skew}(\vec{\omega}) + \frac{1}{2}\mathrm{Skew}(\vec{\omega})^2 \cdots$$

Gradient and Hessian Including Rotation

the only changing term

$$W(R) = \|R\vec{p} - \vec{q}\|^2$$

$$= (R\vec{p} - \vec{q})^T(R\vec{p} - \vec{q}) = \|\vec{p}\|^2 - 2\vec{q}^T R\vec{p} + \|\vec{q}\|^2$$

$$-2\vec{q}^T R(\vec{\omega})\vec{p} = -2\vec{q}^T \exp\{\mathrm{Skew}(\vec{\omega})\} R\vec{p}$$

2nd order

$$\cong -2\vec{q}^T \left\{ I + \mathrm{Skew}(\vec{\omega}) + \frac{1}{2}\mathrm{Skew}(\vec{\omega})^2 \right\} R\vec{p}$$

1st order

$$-2\vec{q}^T \mathrm{Skew}(\vec{\omega}) R\vec{p} = -2\vec{\omega}^T \mathrm{Skew}(R\vec{p})\vec{q} = \vec{\omega}^T(2\vec{q}\times R\vec{p})$$

$$-\vec{q}^T \mathrm{Skew}(\vec{\omega})^2 R\vec{p} = -\vec{q}^T[\vec{\omega}\otimes\vec{\omega} - \vec{\omega}^T\vec{\omega}I]R\vec{p} = \vec{\omega}^T[\vec{q}^T R\vec{p}I - \vec{q}\otimes R\vec{p}]\vec{\omega}$$

Gradient and Hessian Including Rotation

$$W\left(R(\vec{\omega})\right) = \|R(\vec{\omega})\vec{p} - \vec{q}\|^2$$

Gradient: $\dfrac{\partial W}{\partial \vec{\omega}} = \dfrac{\partial}{\partial \vec{\omega}}\{\vec{\omega}^T(2\vec{q}\times R\vec{p})\} = 2\vec{q}\times R\vec{p}$

Hessian: $\dfrac{\partial^2 W}{\partial \vec{\omega}^2} = \dfrac{\partial^2}{\partial \vec{\omega}^2}\{\vec{\omega}^T(\vec{q}^T R\vec{p}I - \vec{q}\otimes R\vec{p})\vec{\omega}\}$

$$= 2\vec{q}^T R\vec{p}I - \vec{q}\otimes R\vec{p} - R\vec{p}\otimes\vec{q}$$

Hessian must be symmetric!

Differentiation w.r.t Vectors

- Transform the equation into a polynomial

$$W(\vec{\omega}) = \cdots$$
$$= \cdots$$
$$= a + \vec{b}^T\vec{\omega} + \vec{\omega}^T C \vec{\omega} + \cdots$$

Gradient: $\dfrac{\partial W}{\partial \vec{\omega}} = \vec{b}$

Hessian: $\dfrac{\partial^2 W}{\partial \vec{\omega}^2} = C^T + C$