Time Integration

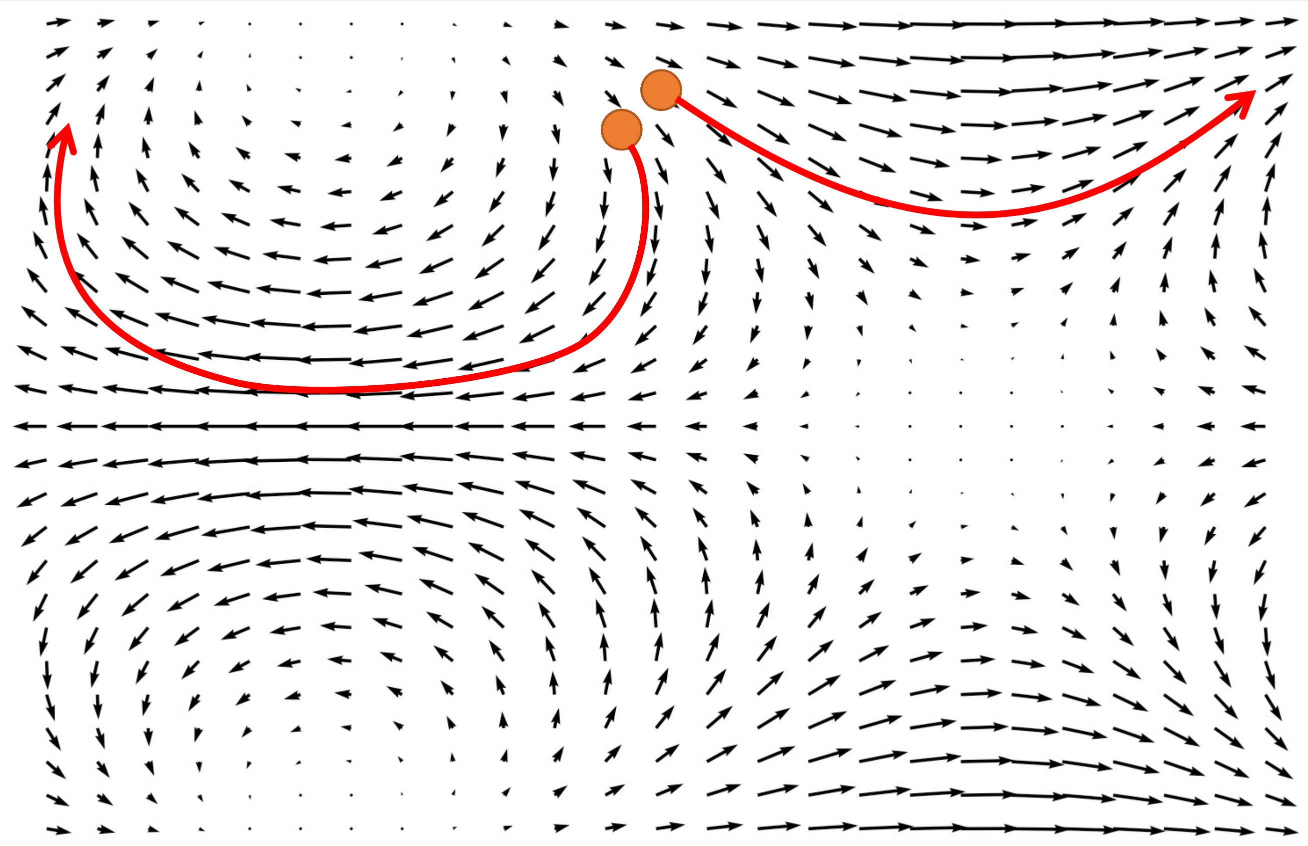
時間積分

System of Differential Equations

連立線形微分方程式

Tracing a Particle in a Velocity Field

- E.g., massless particle in a steady flow



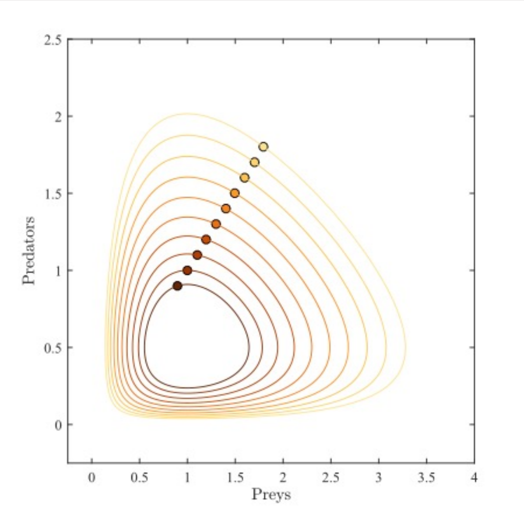
System of 1st Order Differential Equations

- Moving a particle inside a vector field
  - Electrical engineering
  - Control theory
  - System biology

$$\frac{d\vec{x}}{dt} = f(\vec{x})$$

Lotka–Volterra equations
(a.k.a predetors/preys equation)

$$\frac{dx}{dt} = \alpha x - \beta xy$$
$$\frac{dy}{dt} = \delta xy - \gamma y$$



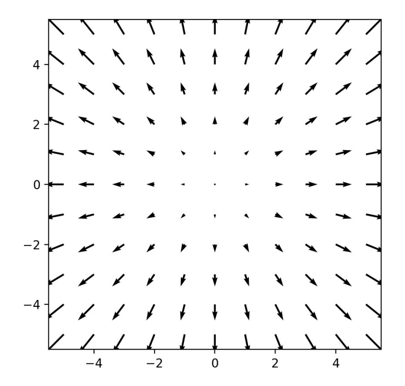
(Wikipedia: Lotka–Volterra equations

Linear 1st Oder System of Diff. Eqn.

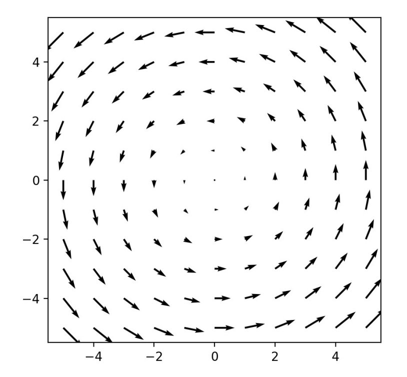
- What if $f(\vec{x})$ is linear?

$$\frac{d\vec{x}}{dt} = A\vec{x}$$

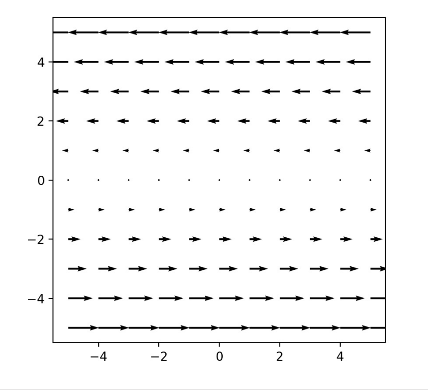
$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$



$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$



$$A = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$$



Solution of Differential Equations

1st order differential equation

$$\frac{dx}{dt} = ax \xrightarrow{\text{solution}} x(t) = e^{at}x(0)$$

System of 1st order differential equations

$$\frac{d\vec{x}}{dt} = A\vec{x} \xrightarrow{\text{solution}} \vec{x}(t) = e^{At}\vec{x}(0)$$

Matrix Exponential

- The Taylor expansion of the exponential function

$$e^{x} = 1 + \frac{1}{1!}x + \frac{1}{2!}x^{2} + \frac{1}{3!}x^{3} + \cdots$$

$$e^{At} = E + \frac{1}{1!}At + \frac{1}{2!}(At)^{2} + \frac{1}{3!}(At)^{3} + \cdots$$

$$\frac{d}{dt}\left(e^{At}\right) = ?$$

check it out!

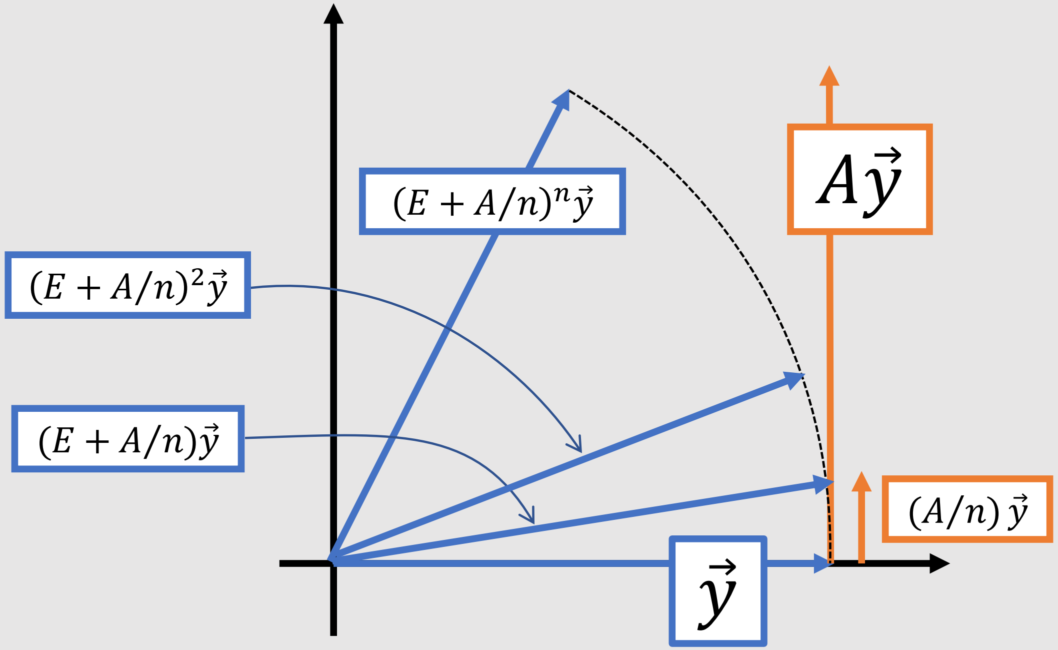
Geometrical Interpretation

- Let's go back to the definition of the exponential

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n \xrightarrow{\text{multi-variable}} e^A = \lim_{n\to\infty}\left(E+\frac{A}{n}\right)^n$$

For example, let A is a matrix to compute tangent in 2D

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$



Compound Interest(複利効果)

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}$$

$$\$1 \xrightarrow[\text{(12months)}]{100\%} \$2$$

$$\$1 \xrightarrow[\text{(6 months)}]{50\%} \$1.5 \xrightarrow[\text{(6 months)}]{50\%} \$2$$

$$\$1 \xrightarrow[\text{(3 months)}]{25\%} \$1.25 \xrightarrow[\text{(3 months)}]{25\%} \$1.5625 \xrightarrow[\text{(3 months)}]{25\%} \$1.95 \xrightarrow[\text{(3 months)}]{25\%} \$2.4375$$

$$\$1 \xrightarrow[\text{(1day)}]{1/365\%} \$..... \xrightarrow[\text{(1day)}]{1/365\%} \$..... \longrightarrow \quad \longrightarrow \$2.714..$$

$$\$1 \rightarrow \$..... \rightarrow \$..... \rightarrow \$..... \rightarrow \rightarrow \rightarrow \$e(2.718..)$$

Wealth is Exponential

Compound interest is the 8th wonder of the world. He who understands it, earns it; he who doesn't, pays it.

-Albert Einstein

r>g

(i.e, you can earn more from investment than working hard)

Thomas Piketty, Capital in the Twenty-First Century

(Wikipedia)

Diagonalization and Matrix Exponential

$$Av_i = \lambda_i v_i \xrightarrow{\text{eigen decomposition}} A = V\Lambda V^{-1}$$

$$e^{At} = E + \frac{1}{1!}At + \frac{1}{2!}(At)^2 + \frac{1}{3!}(At)^3 + \cdots$$

check it out!

System of 2nd Order Differential Equation

- 2nd order system can be transformed into a 1st order system

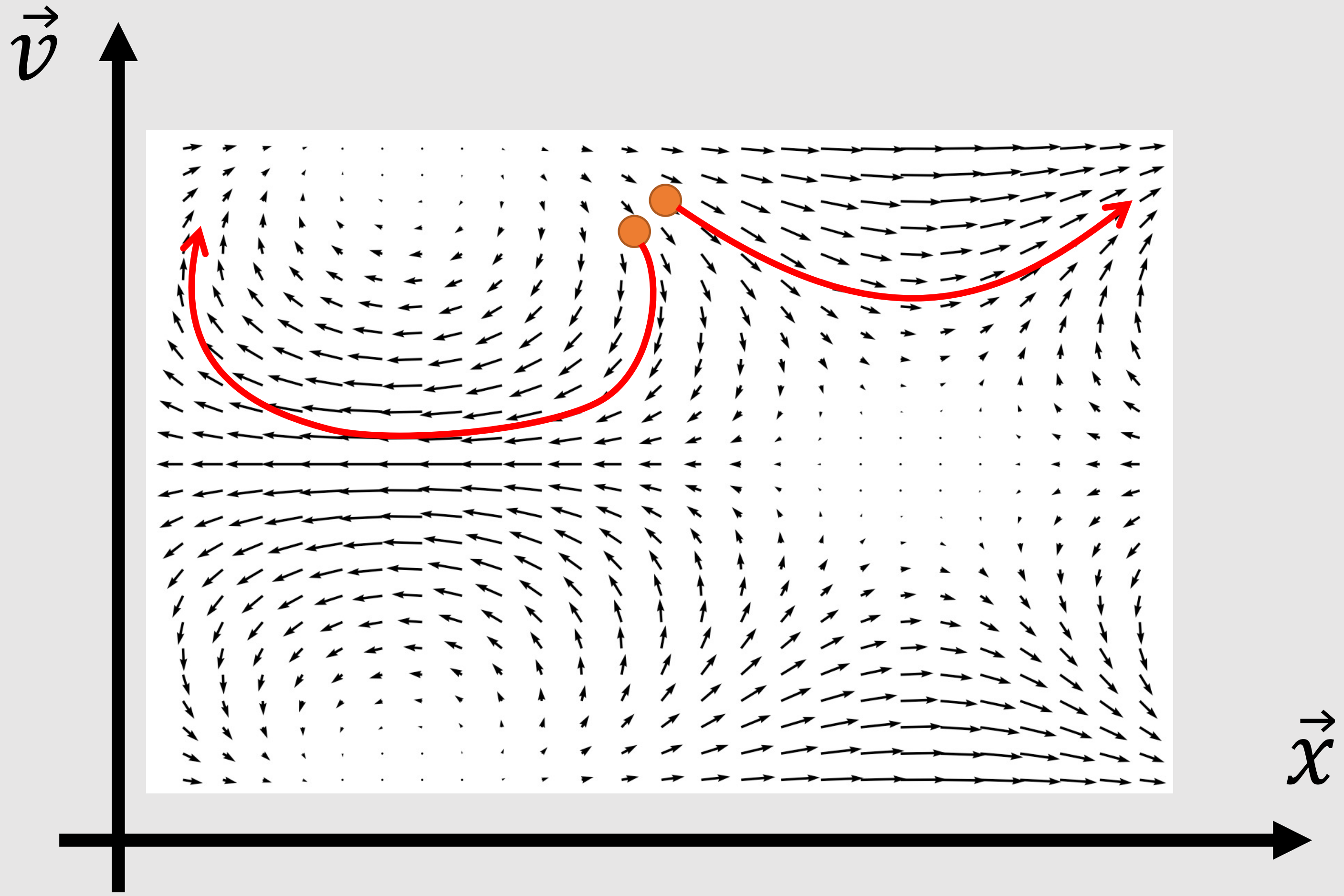
$$\frac{d^2\vec{x}}{dt^2} - A\frac{d\vec{x}}{dt} - \mathrm{B}\vec{x} = 0$$

$$\Big\downarrow \quad \boxed{\vec{v} = \frac{d\vec{x}}{dt}}$$

$$\frac{d}{dt}\begin{pmatrix} \vec{v} \\ \vec{x} \end{pmatrix} = \begin{bmatrix} A & B \\ E & 0 \end{bmatrix}\begin{pmatrix} \vec{v} \\ \vec{x} \end{pmatrix}$$

Analyzing stability of this system requires Laplace transformation, which is beyond the scope of this lecture

Mechanics: Trajectory in Phase Space

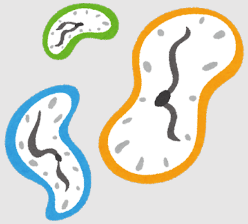


Discrete Time Integration

Why Temporal Discretization?

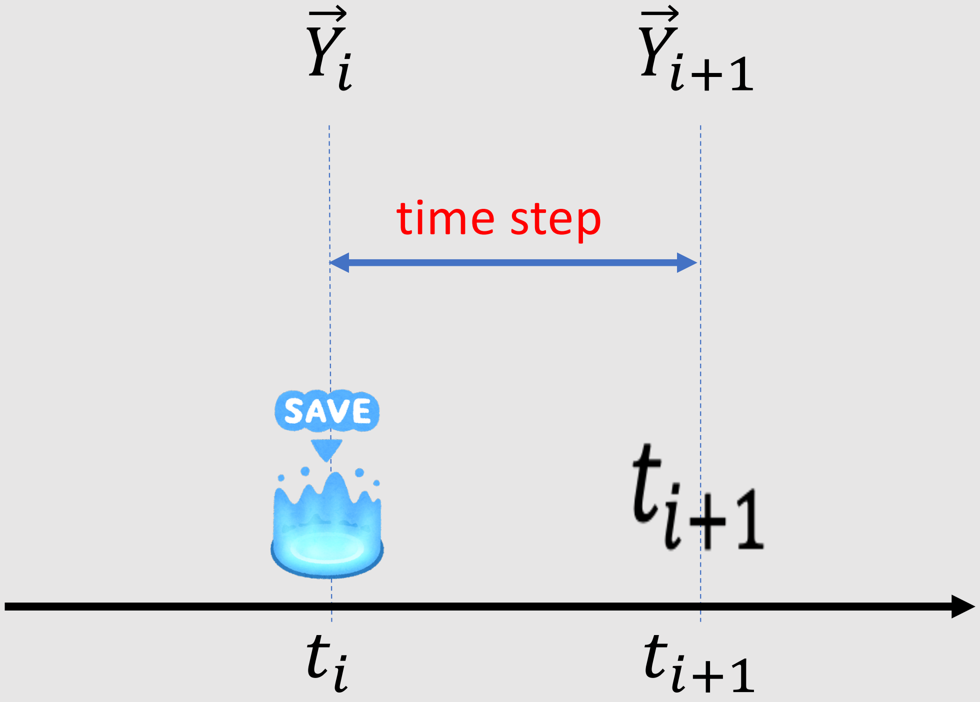
- Dynamic system doesn’t always have an analytical solution

- Computer cannot handle continuous value
  - Similar to “quantization” and “sampling” in audio processing



Time Integration for Temporal Discretization

- The interval is called “time step”



$\vec{Y}_i$ $\vec{Y}_{i+1}$

time step

t_{i+1}

t_i t_{i+1}

Recurrent formula

$$\vec{Y}_{i+1} = F(\vec{Y}_i)$$

Given equation of motion,
what are the $\vec{Y}_i$ and $F(\;)$?

Approximating Gradient by Difference

$$\boxed{\frac{dx}{dt} = F(x)}$$

forward(explicit)
Euler method

$$\frac{x_{i+1} - x_i}{dt} = F(x_i)$$

Simple but
Unstable

backward(implicit)
Euler method

$$\frac{x_{i+1} - x_i}{dt} = F(x_{i+1})$$

Complicated but
Stable

Recurrence Relation from Backward Euler

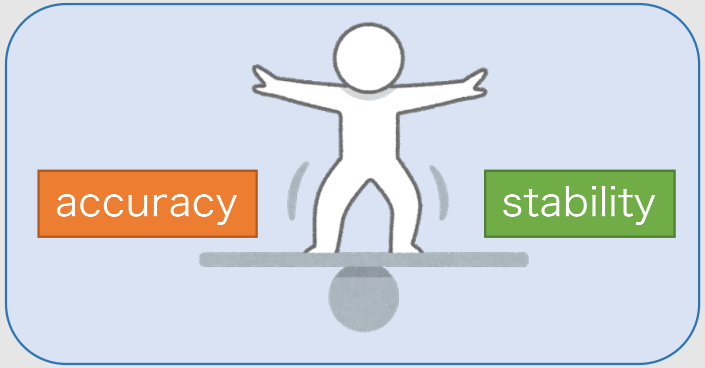
$$\frac{x_{i+1} - x_i}{dt} = F(x_{i+1})$$

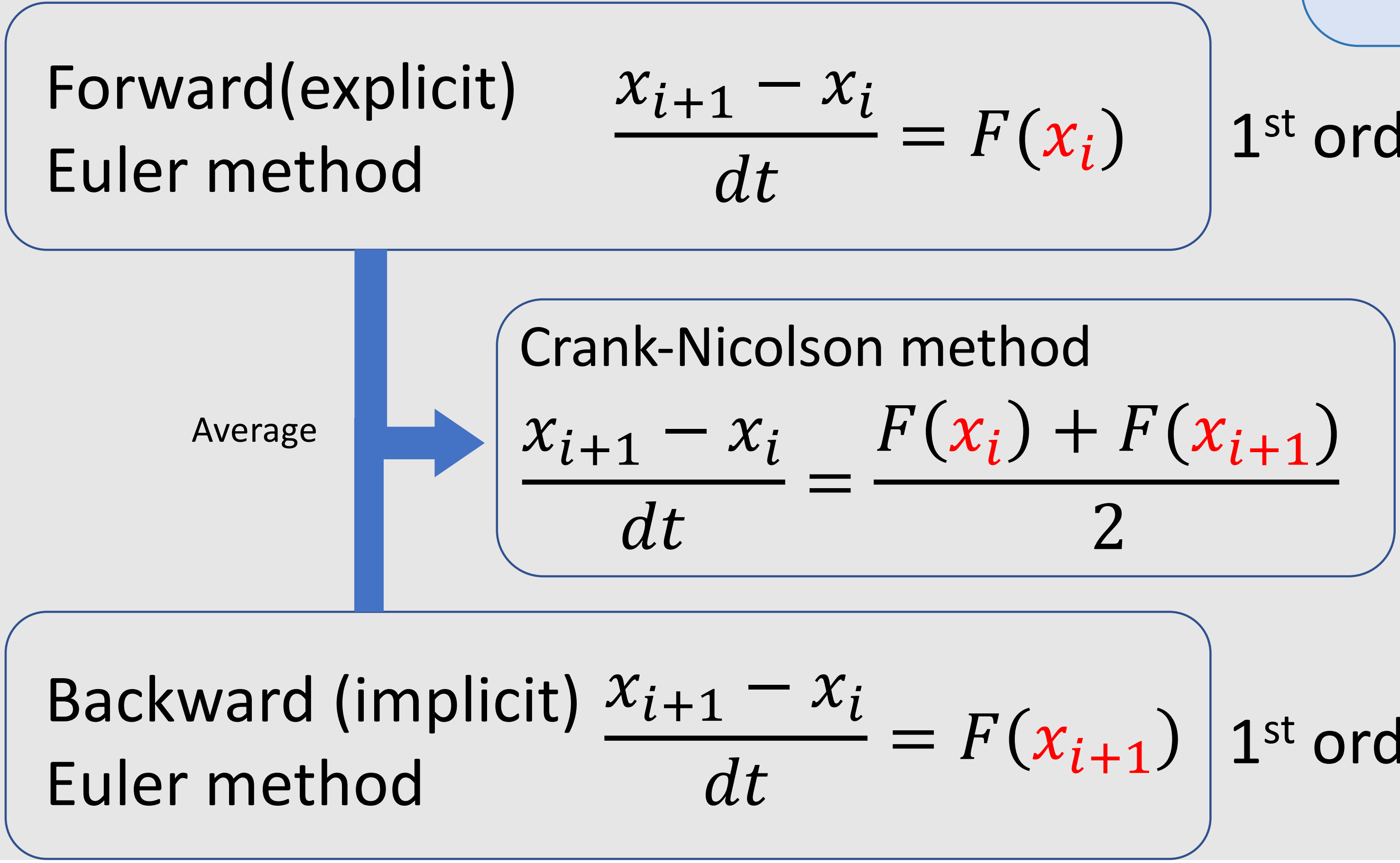
Taylor's expansion

$$\cong f(x_i) + \left.\frac{dF}{dx}\right|_{x_i} (x_{i+1} - x_i)$$

(write equation Here)

Accuracy of Time Integration





Forward(explicit)
Euler method

$$\frac{x_{i+1} - x_i}{dt} = F(x_i)$$

1st order

Average

Crank-Nicolson method

$$\frac{x_{i+1} - x_i}{dt} = \frac{F(x_i) + F(x_{i+1})}{2}$$

2nd order

Backward (implicit)
Euler method

$$\frac{x_{i+1} - x_i}{dt} = F(x_{i+1})$$

1st order

2nd-order Differential Eqn. by Backward Euler

Backward
Euler method

$$\frac{ds}{dt} = \frac{s_{i+1} - s_i}{dt} = F(s_{i+1})$$

plug in $s_i = \begin{pmatrix} \vec{v}_\mathrm{i} \\ \vec{x}_\mathrm{i} \end{pmatrix}$

(write equations here)

Simple Example: Particle Under Gravity

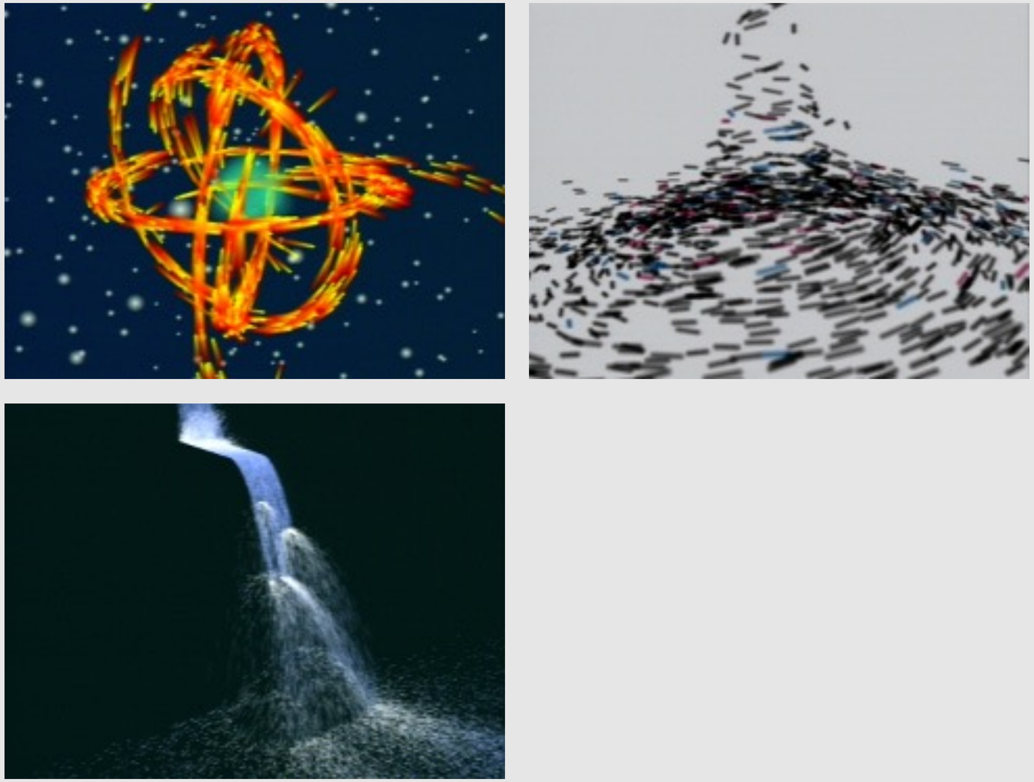
$$m\vec{a} = m\vec{g} \Rightarrow \begin{cases} \vec{v}_{i+1} - \vec{v}_i = dt \cdot \vec{a}_{i+1} \\ \vec{x}_{i+1} - \vec{x}_i = dt \cdot \vec{v}_{i+1} \end{cases}$$

$$\Downarrow \quad \vec{a} = \vec{g}$$

$$\boxed{\begin{aligned} \vec{v}_{i+1} &= \vec{v}_i + dt \cdot \vec{g} \\ \vec{x}_{i+1} &= \vec{x}_i + dt \cdot (\vec{v}_i + dt \cdot \vec{g}) \end{aligned}}$$

Karl Sim’s Particle Dreams, 1988

https://www.karlsims.com/particle-dreams.html

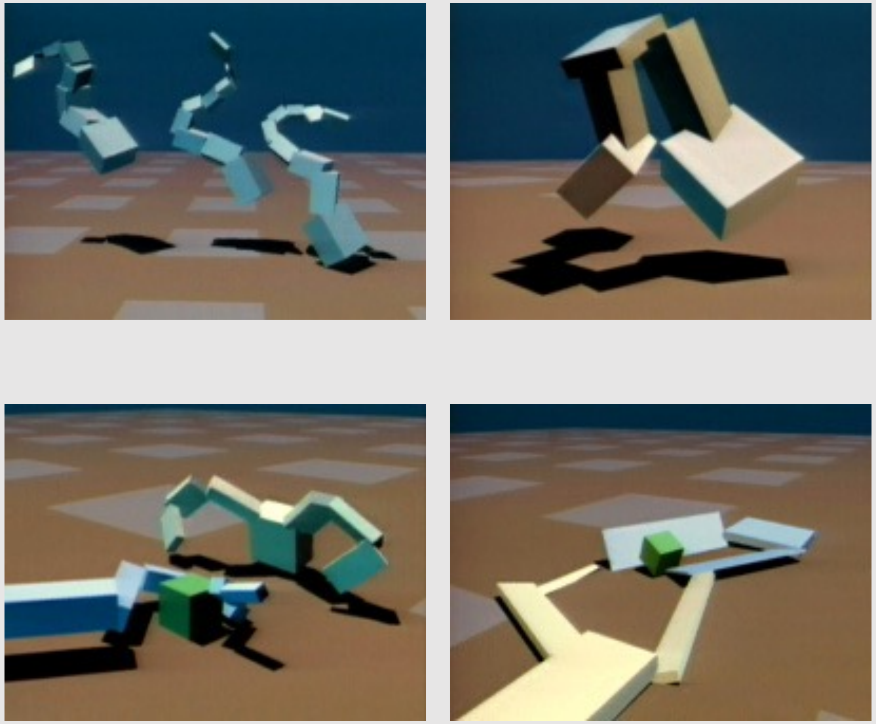


https://www.youtube.com/watch?v=5QEp-oPaQto

Karl Sim’s Another Awesome Work

K.Sims, “Evolved Virtual Creatures ”, Siggraph ‘94

https://www.karlsims.com/evolved-virtual-creatures.html



https://www.youtube.com/watch?v=RZtZia4ZkX8

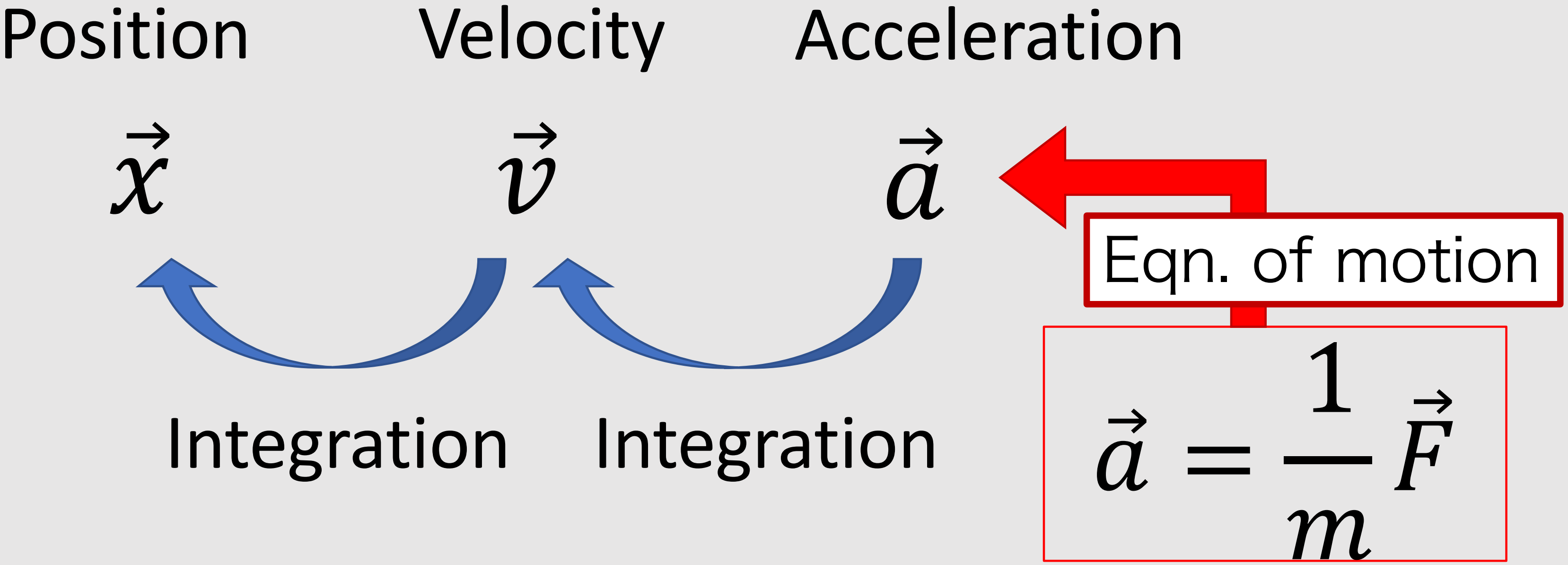
Advanced Topics

- Runge-Kutta method
- Variational Implicit Euler Method
- Symplectic Integrator
- Lie group integrator

End

Time Integration

Recurrence formula from equation of motion



Position Velocity Acceleration

$\vec{x}$ $\vec{v}$ $\vec{a}$

Integration Integration

Eqn. of motion

$$\vec{a} = \frac{1}{m}\vec{F}$$

Time Integration: 1st-order Differential Eqn.

- Given $\vec{x}_i$, solve for $\vec{x}_{i+1}$

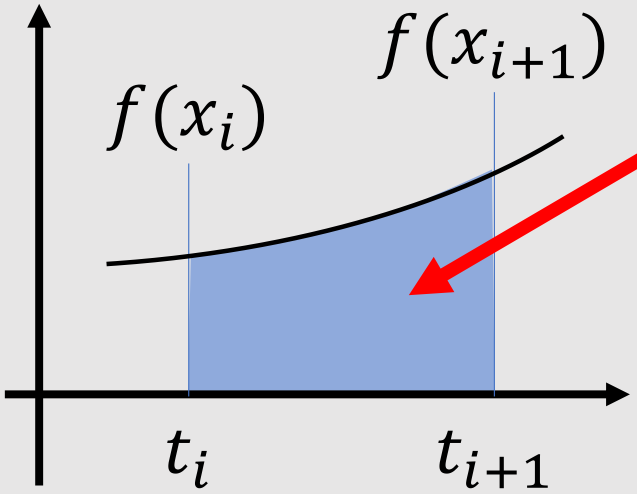
$$\frac{dx}{dt} = f(x) \quad \xrightarrow{\text{Integration}} \quad x_{i+1} = x_i + \int_{t_i}^{t_{i+1}} f(x)\, dt$$

（ここに手書きで式を書く）

Time Integration: 1st-order Differential Eqn.

- Compute $\vec{x}_{i+1}$ when $\vec{x}_i$ is given

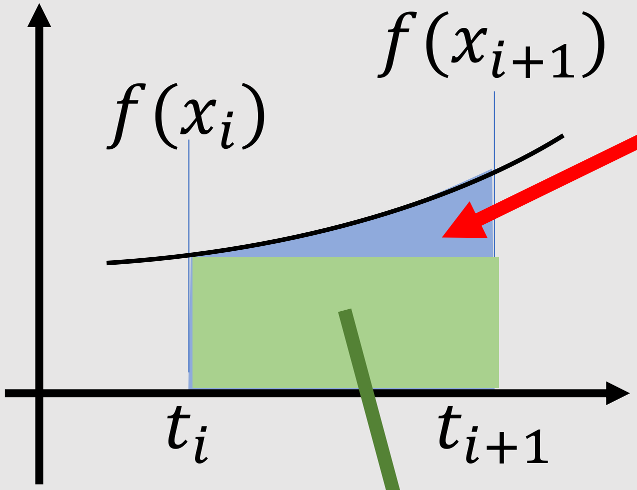
$$\frac{dx}{dt} = f(x) \xrightarrow{\text{integration}} x_{i+1} = x_i + \int_{t_i}^{t_{i+1}} f(x)\,dt$$



Area here makes
$x_{i+1} - x_i$

$f(x)$ is unknown and x is unknown
How the area is approximated

Time Integration: 1st-order Differential Eqn.



Area here makes $x_{i+1} - x_i$

$f(x)$ is unknown because x is unknown.
How area can be approximated

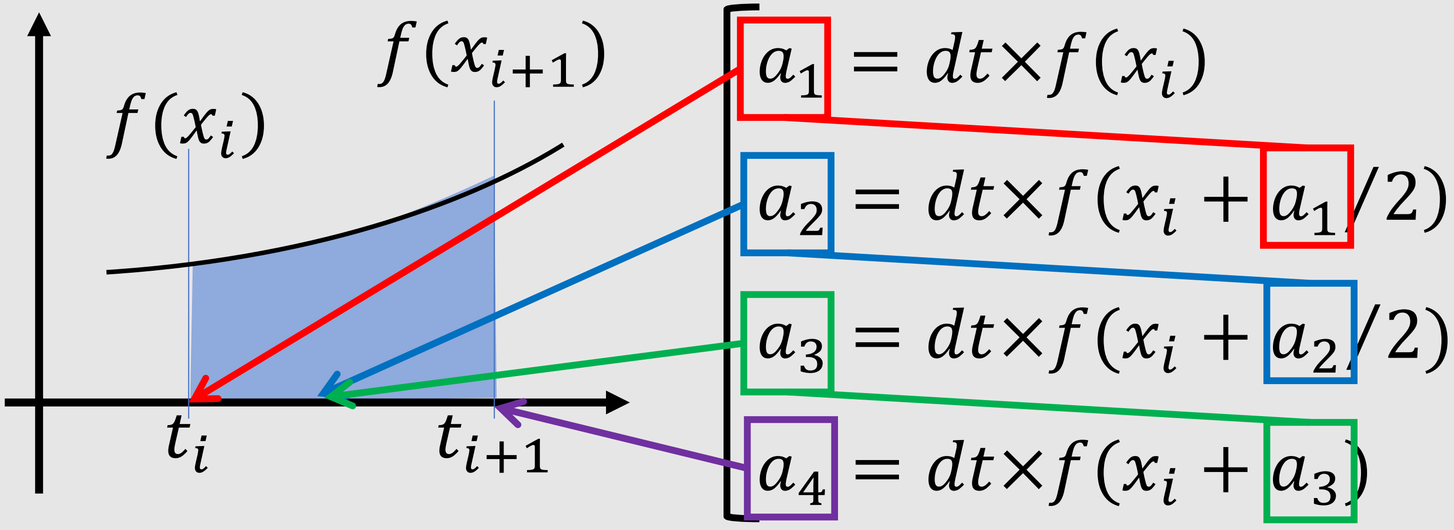
Advanced approach
Runge-Kutta method

Area：$dt\times f(x_i)$

Forward(explicit)
Euler method $x_{i+1} = x_i + dt\times f(x_i)$

Runge-Kutta Method (4th order)

- Approximating are $a = x_{i+1}$- x_i with 4 different ways



$$\boxed{a_1} = dt \times f(x_i)$$

$$\boxed{a_2} = dt \times f(x_i + \boxed{a_1}/2)$$

$$\boxed{a_3} = dt \times f(x_i + \boxed{a_2}/2)$$

$$\boxed{a_4} = dt \times f(x_i + \boxed{a_3})$$

averaging

$$x_{i+1} = x_i + \frac{1}{6}(a_1 + 2a_1 + 2a_1 + a_1)$$