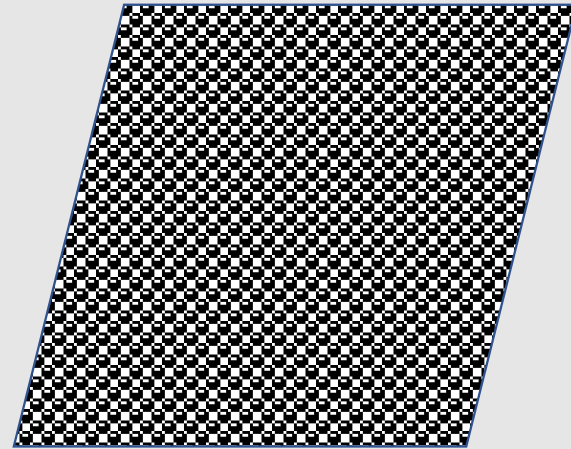
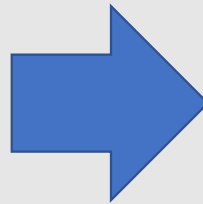
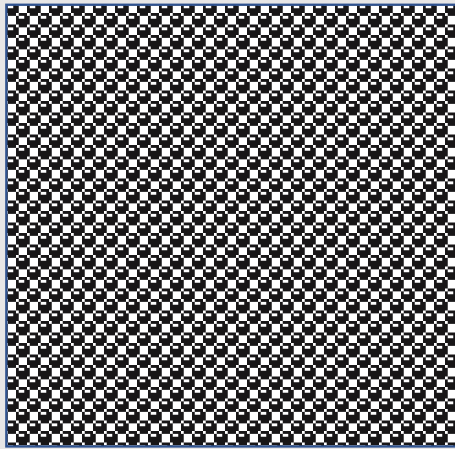


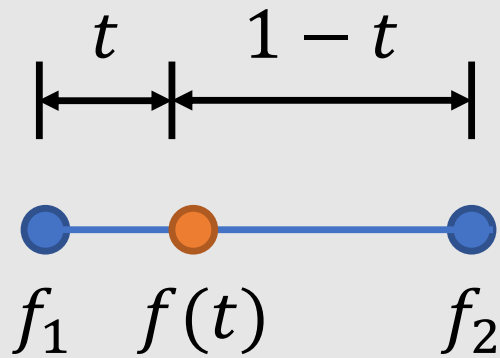
Grid & Mesh Interpolation

Continuum Approximation

- Drastically reducing degrees of freedom (DoFs)



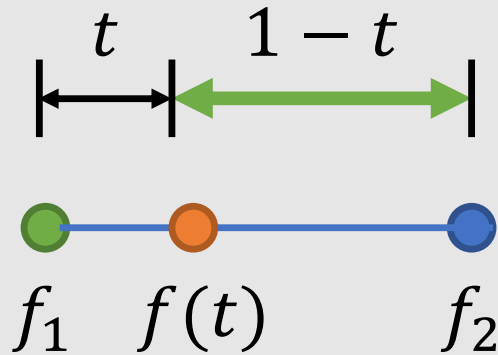
Linear Interpolation (1D)



$$f(0) = f_1$$

$$f(1) = f_2$$

Linear Interpolation (1D)



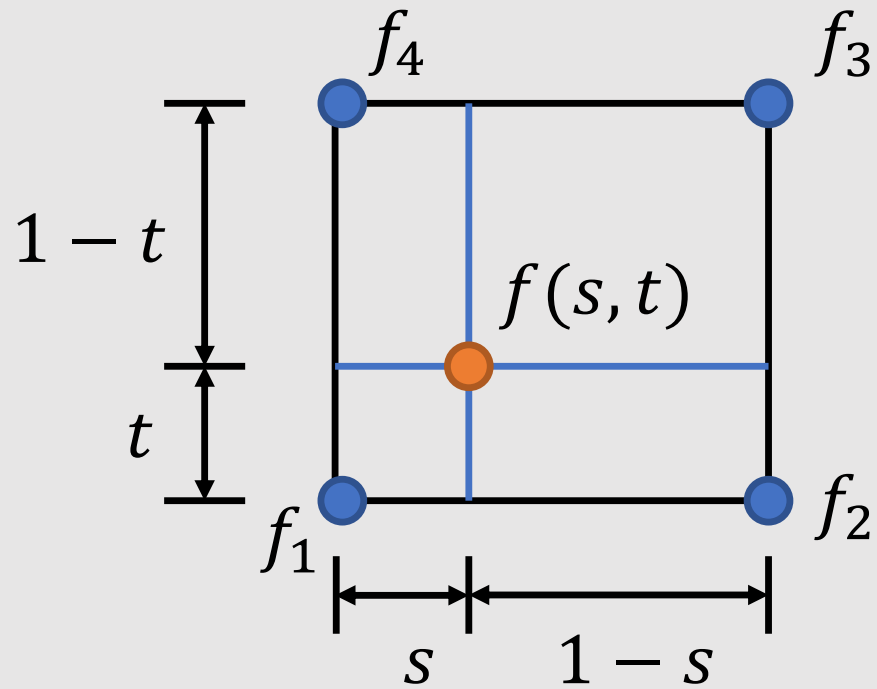
$$f(0) = f_1$$

$$f(1) = f_2$$

Weights sums to 1

$$f(t) = (1 - t)f_1 + tf_2$$

Bilinear Interpolation (2D)



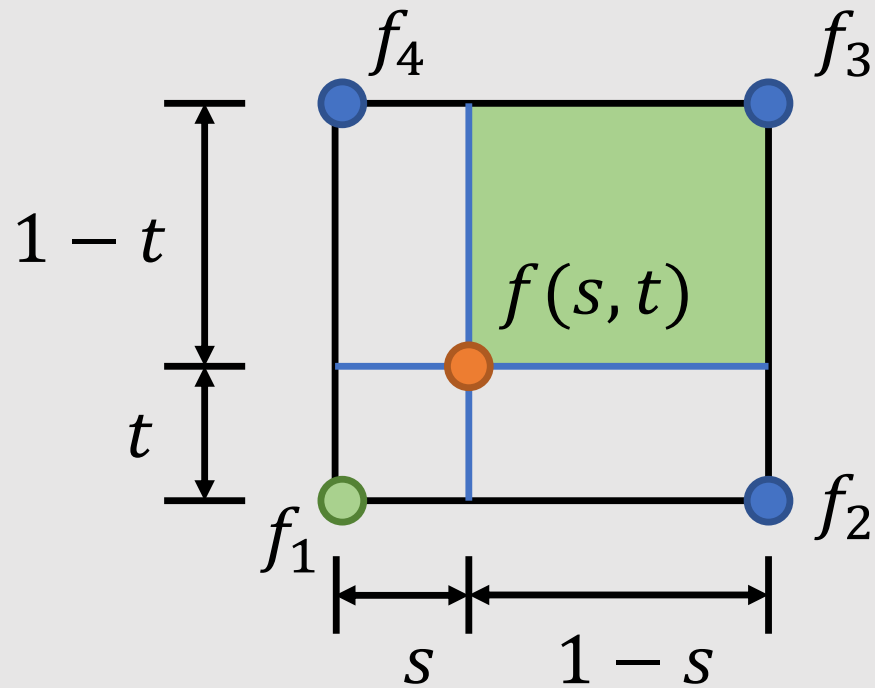
$$f(0,0) = f_1$$

$$f(1,0) = f_2$$

$$f(1,1) = f_3$$

$$f(0,1) = f_4$$

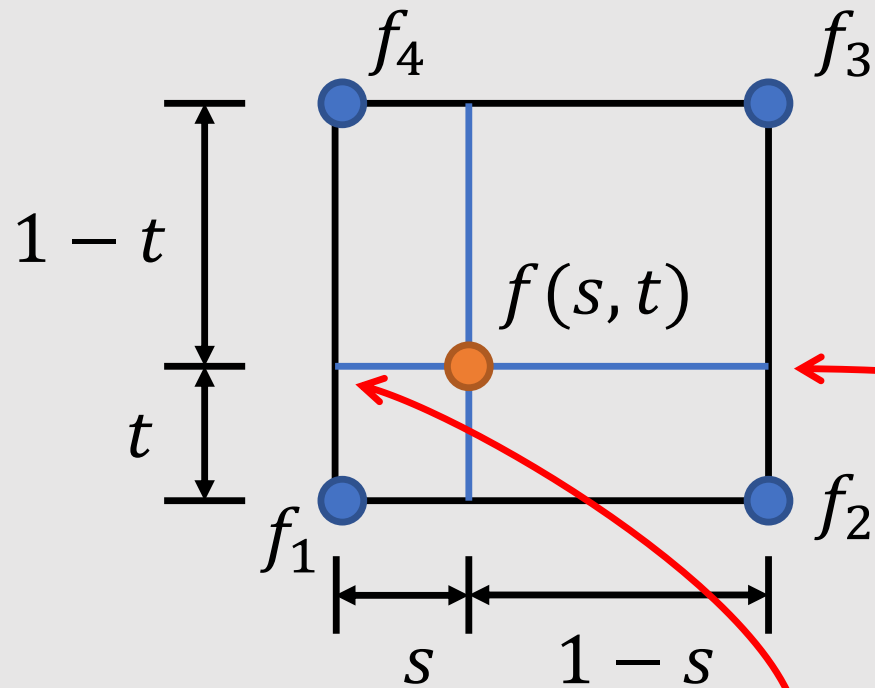
Bilinear Interpolation (2D)



Weights sums to 1

$$f(s, t) = (1-s)(1-t)f_1 + s(1-t)f_2 + stf_3 + (1-s)tf_4$$

Bilinear Interpolation (2D)

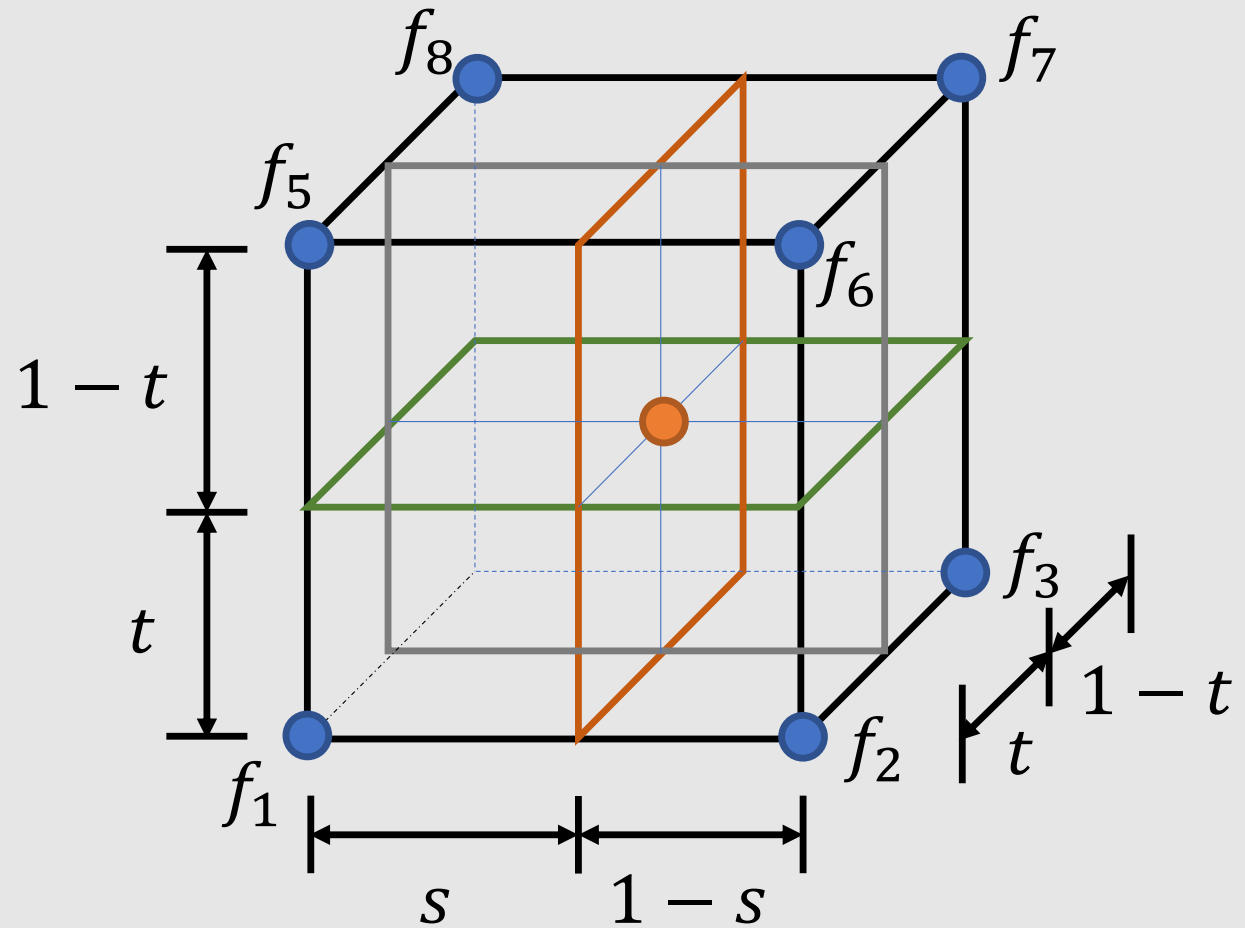


$f(s, t)$ is Linear in horizontal & vertical directions (not in an oblique direction)

$$f(s, t) = (1 - s)[(1 - t)f_1 + tf_4] + s[(1 - t)f_2 + tf_3]$$

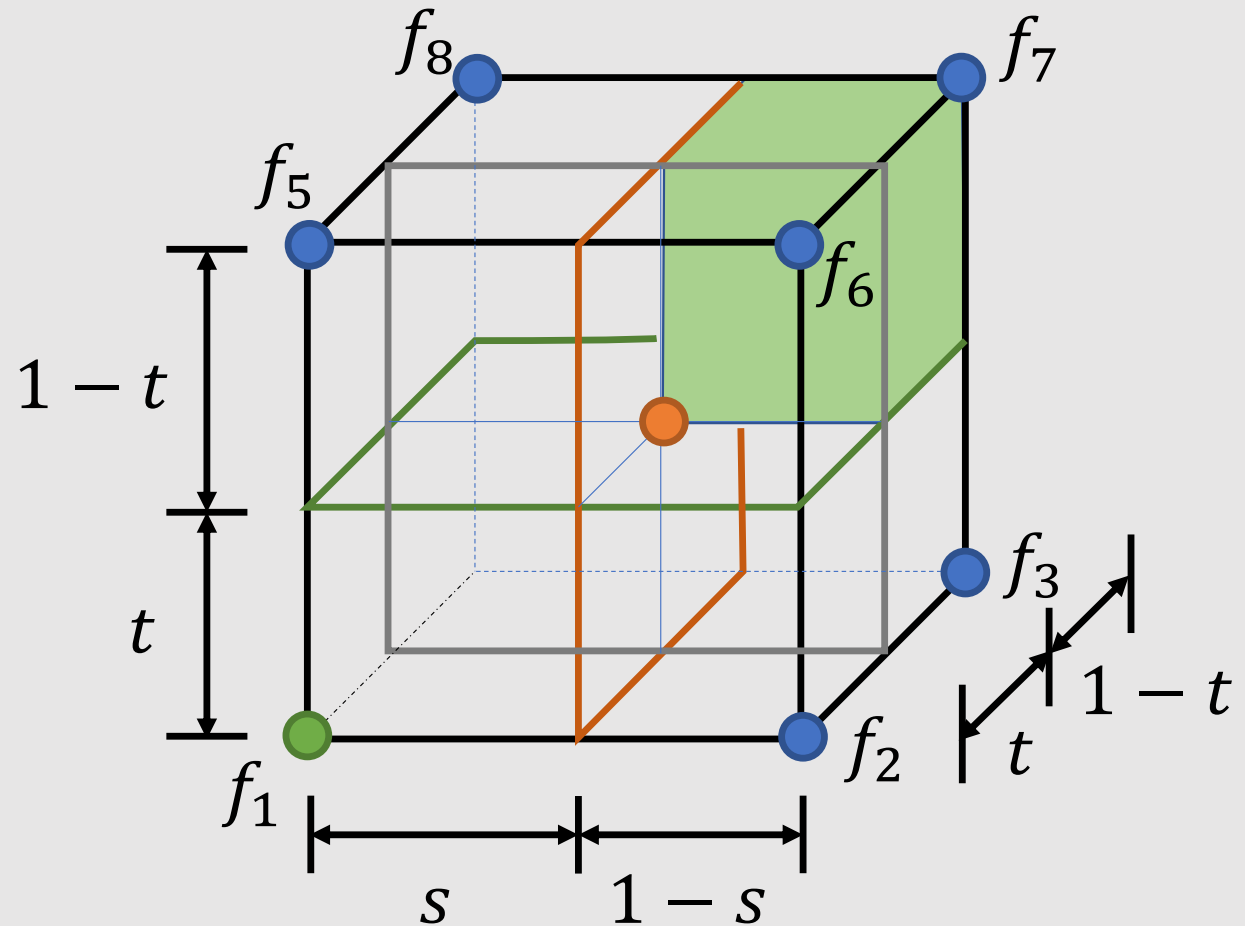
Trilinear Interpolation (3D)

$$\begin{aligned} f(s, t, u) = & (1-s)(1-t)(1-u)f_1 + \\ & s(1-t)(1-u)f_2 + \\ & st(1-u)f_3 + \\ & (1-s)t(1-u)f_4 + \\ & (1-s)(1-t)uf_5 + \\ & s(1-t)uf_6 + \\ & stuf_7 + \\ & (1-s)tu f_8 \end{aligned}$$



Trilinear Interpolation (3D)

$$\begin{aligned} f(s, t, u) = & (1-s)(1-t)(1-u)f_1 + \\ & s(1-t)(1-u)f_2 + \\ & st(1-u)f_3 + \\ & (1-s)t(1-u)f_4 + \\ & (1-s)(1-t)uf_5 + \\ & s(1-t)uf_6 + \\ & stuf_7 + \\ & (1-s)tu f_8 \end{aligned}$$



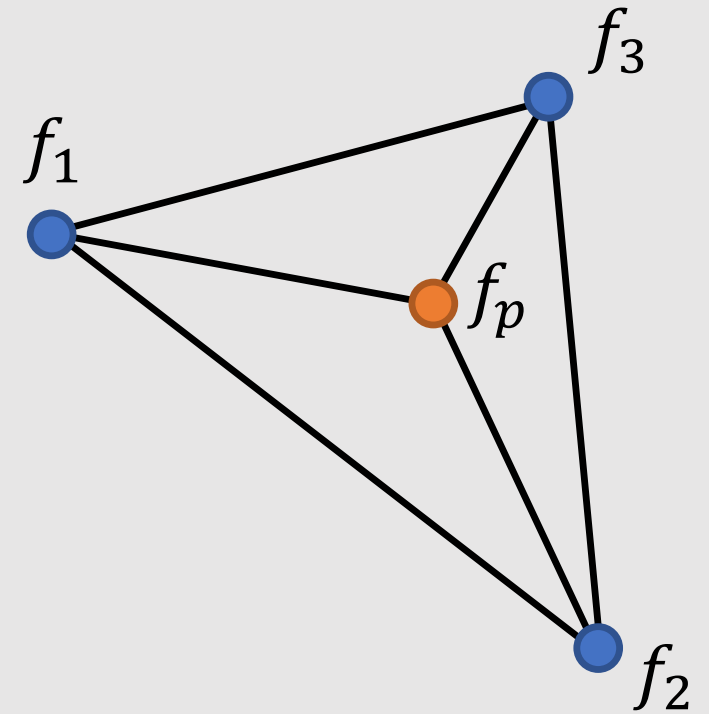
Barycentric Coordinates for a Triangle

$$f_p = f(L_1, L_2, L_3) = L_1 f_1 + L_2 f_2 + L_3 f_3$$

$$L_1 = \frac{\text{Area}(p, 2, 3)}{\text{Area}(1, 2, 3)}$$

$$L_2 = \frac{\text{Area}(1, p, 3)}{\text{Area}(1, 2, 3)}$$

$$L_3 = \frac{\text{Area}(1, 2, p)}{\text{Area}(1, 2, 3)}$$



Barycentric Coordinates for a Tetrahedron

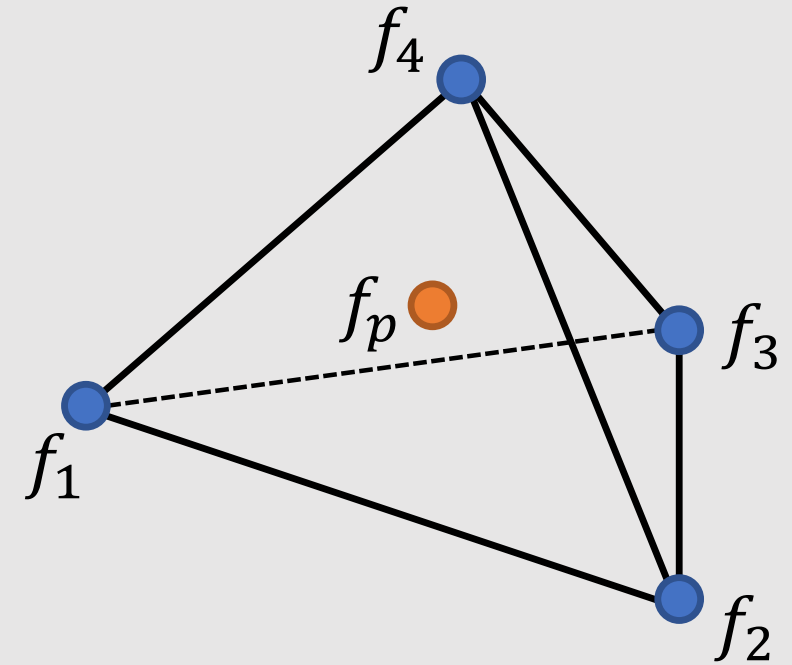
$$\begin{aligned} f_p &= f(L_1, L_2, L_3, L_4) \\ &= L_1 f_1 + L_2 f_2 + L_3 f_3 + L_4 f_4 \end{aligned}$$

$$L_1 = \frac{\text{Volume}(p, 2, 3, 4)}{\text{Volume}(1, 2, 3, 4)}$$

$$L_2 = \frac{\text{Volume}(1, p, 3, 4)}{\text{Volume}(1, 2, 3, 4)}$$

$$L_3 = \frac{\text{Volume}(1, 2, p, 4)}{\text{Volume}(1, 2, 3, 4)}$$

$$L_4 = \frac{\text{Volume}(1, 2, 3, p)}{\text{Volume}(1, 2, 3, 4)}$$



Barycentric Coordinates for a Tetrahedron

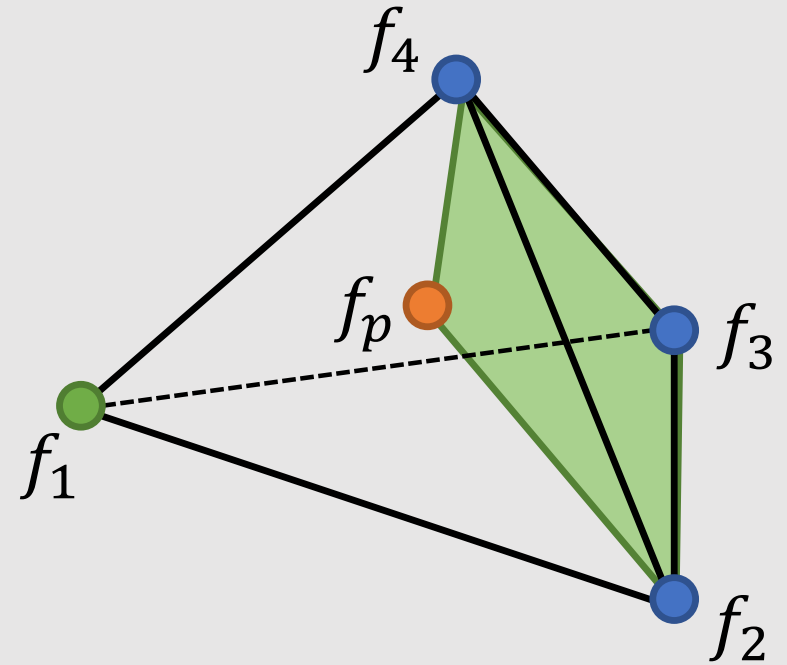
$$\begin{aligned} f_p &= f(L_1, L_2, L_3, L_4) \\ &= L_1 f_1 + L_2 f_2 + L_3 f_3 + L_4 f_4 \end{aligned}$$

$$L_1 = \frac{\text{Volume}(p, 2, 3, 4)}{\text{Volume}(1, 2, 3, 4)}$$

$$L_2 = \frac{\text{Volume}(1, p, 3, 4)}{\text{Volume}(1, 2, 3, 4)}$$

$$L_3 = \frac{\text{Volume}(1, 2, p, 4)}{\text{Volume}(1, 2, 3, 4)}$$

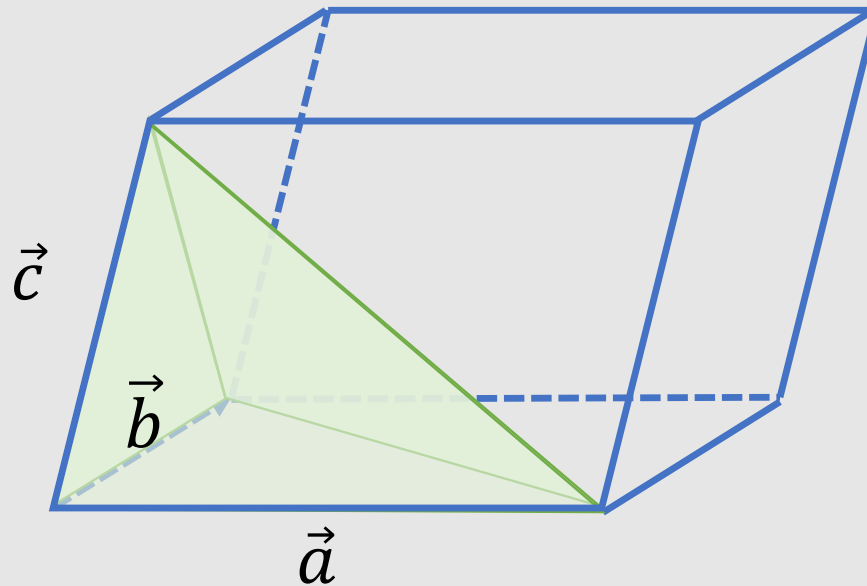
$$L_4 = \frac{\text{Volume}(1, 2, 3, p)}{\text{Volume}(1, 2, 3, 4)}$$



Differentiation of Interpolated Values

Volume of Tetrahedron from Parallelepiped

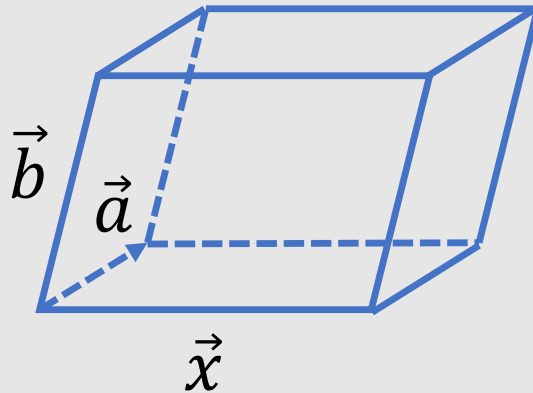
- Volume of parallelepiped: $V = \vec{a} \cdot (\vec{b} \times \vec{c})$
- Volume of tetrahedron: $V = 1/6 \vec{a} \cdot (\vec{b} \times \vec{c})$



Differential of Scalar Triple Product

$$\begin{aligned} W(\vec{x}) &= \vec{b} \cdot (\vec{a} \times \vec{x}) \\ &= \vec{x} \cdot (\vec{b} \times \vec{a}) \end{aligned}$$

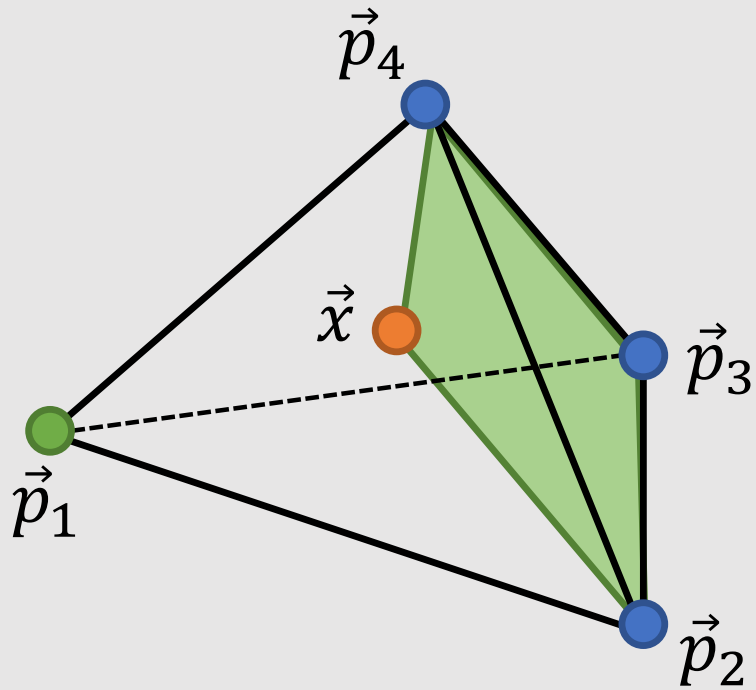
$$\frac{\partial W}{\partial \vec{x}} = \vec{b} \times \vec{a}$$



The volume is **linear**
to the position



Volume of Tetrahedron



$$L_1 = \frac{\text{Volume}(\vec{x}, \vec{p}_2, \vec{p}_3, \vec{p}_4)}{\text{Volume}(\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4)}$$

$$\begin{aligned} \text{Volume}(\vec{x}, \vec{p}_2, \vec{p}_3, \vec{p}_4) \\ = 1/6 (\vec{x} - \vec{p}_2) \cdot \{(\vec{p}_4 - \vec{p}_2) \times (\vec{p}_3 - \vec{p}_2)\} \end{aligned}$$

$$\frac{\partial L_1}{\partial \vec{x}} = \frac{1/6 (\vec{p}_4 - \vec{p}_2) \times (\vec{p}_3 - \vec{p}_2)}{\text{Volume}(\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4)}$$

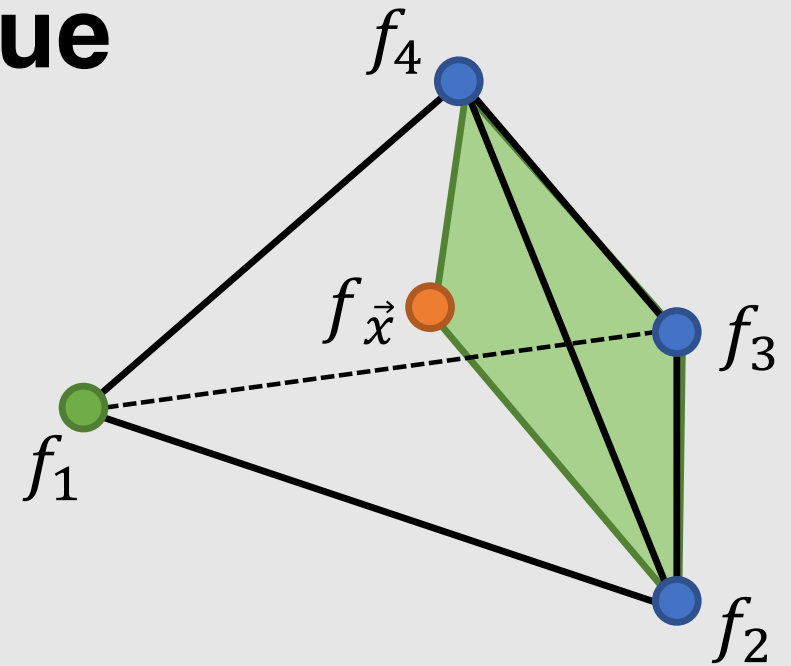
Gradient of Interpolated Value

$$\begin{aligned} f_{\vec{x}} &= f(L_1, L_2, L_3, L_4) \\ &= L_1 f_1 + L_2 f_2 + L_3 f_3 + L_4 f_4 \end{aligned}$$



differentiation

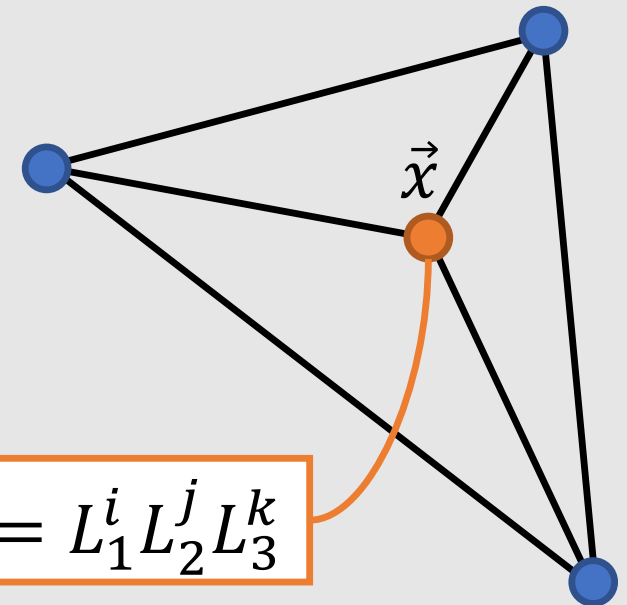
$$\frac{\partial f_{\vec{x}}}{\partial \vec{x}} = \frac{\partial L_1}{\partial \vec{x}} f_1 + \frac{\partial L_2}{\partial \vec{x}} f_2 + \frac{\partial L_3}{\partial \vec{x}} f_3 + \frac{\partial L_4}{\partial \vec{x}} f_4$$



Integration of Polynomial over Simplex

Integration Rule for Triangle

$$\int_{\vec{x} \in \text{Tri}} L_1^i L_2^j L_3^k d\vec{x} = \frac{2! i! j! k! \text{Area}}{(2 + i + j + k)!}$$



Baldoni, Velleda, Nicole Berline, Jesus De Loera, Matthias Köppe, and Michèle Vergne. "How to integrate a polynomial over a simplex." *Mathematics of Computation* 80, no. 273 (2011): 297–325. <https://arxiv.org/abs/0809.2083>

Example of Integration over Triangle

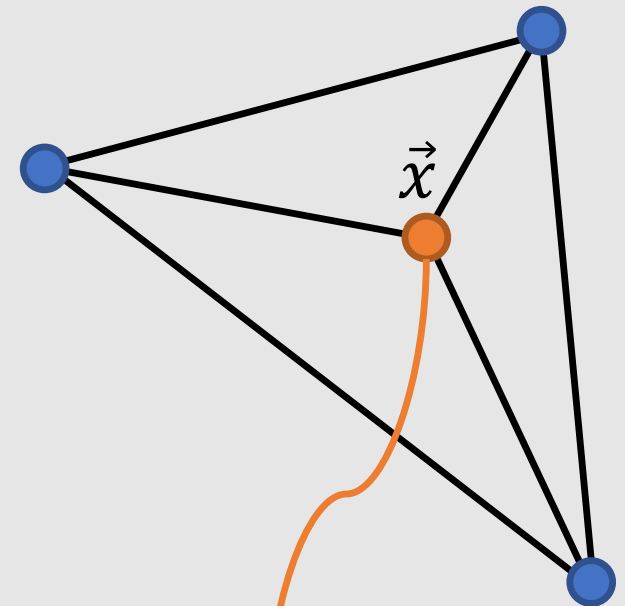
$$\int_{\vec{x} \in Tri} L_1 d\vec{x} = \int_{\vec{x} \in Tri} L_1^1 L_2^0 L_3^0 d\vec{x} = \frac{2! 1! 0! 0! Area}{(2 + 1 + 0 + 0)!} = \frac{1}{3} Area$$

$$\int_{\vec{x} \in Tri} L_1^2 d\vec{x} = \int_{\vec{x} \in Tri} L_1^2 L_2^0 L_3^0 d\vec{x} = \frac{2! 2! 0! 0! Area}{(2 + 2 + 0 + 0)!} = \frac{1}{6} Area$$

$$\int_{\vec{x} \in Tri} L_1 L_2 d\vec{x} = \int_{\vec{x} \in Tri} L_1^1 L_2^1 L_3^0 d\vec{x} = \frac{2! 1! 1! 0! Area}{(2 + 1 + 1 + 0)!} = \frac{1}{12} Area$$

Center of The Gravity of a Triangle in 3D

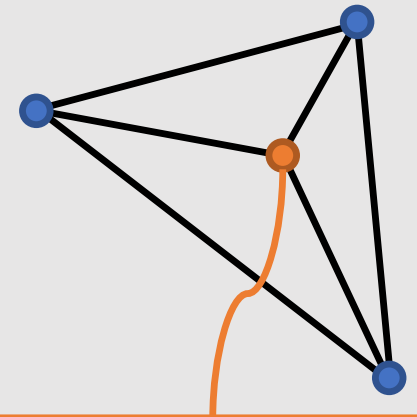
$$\begin{aligned}\vec{p}_{cg} &= \frac{1}{Area} \int_{\vec{x} \in \text{Tri}} \vec{x} d\vec{x} \\ &= \frac{1}{Area} \int_{\vec{x} \in \text{Tri}} L_1 \vec{p}_1 + L_2 \vec{p}_2 + L_3 \vec{p}_3 d\vec{x} \\ &= \frac{1}{3} (\vec{p}_1 + \vec{p}_2 + \vec{p}_3)\end{aligned}$$



$$\vec{x} = L_1 \vec{p}_1 + L_2 \vec{p}_2 + L_3 \vec{p}_3$$

Inertia Tensor of a Triangle in 3D

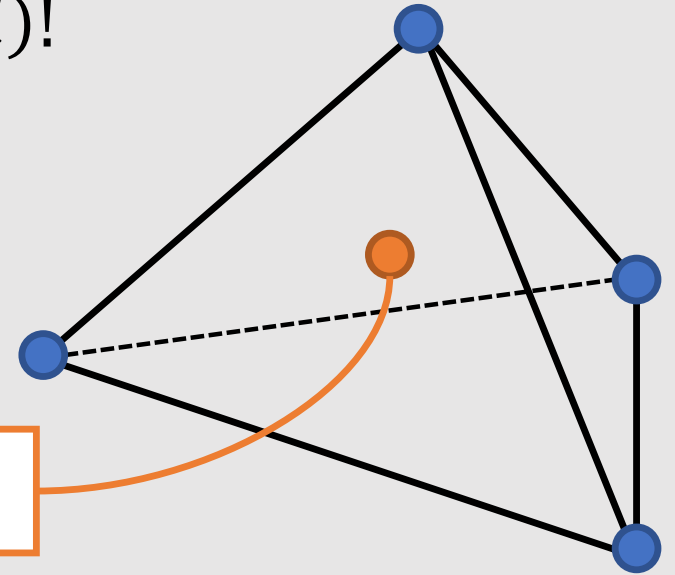
$$\begin{aligned}
 I_{inertia} &= \int_{\vec{x} \in \text{Tri}} -\text{Skew}(\vec{x})\text{Skew}(\vec{x})d\vec{x} \\
 &= \int_{\vec{x} \in \text{Tri}} -\text{Skew}(L_1\vec{p}_1 + L_2\vec{p}_2 + L_3\vec{p}_3)\text{Skew}(L_1\vec{p}_1 + L_2\vec{p}_2 + L_3\vec{p}_3)d\vec{x} \\
 &= \int_{\vec{x} \in \text{Tri}} \sum_{1 \leq i \leq 3} \sum_{1 \leq j \leq 3} -\text{Skew}(L_i\vec{p}_i)\text{Skew}(L_j\vec{p}_j)d\vec{x} \\
 &= \sum_{1 \leq i \leq 3} \sum_{1 \leq j \leq 3} -\text{Skew}(\vec{p}_i)\text{Skew}(\vec{p}_j) \int_{\vec{x} \in \text{Tri}} L_i L_j d\vec{x} \\
 &= \sum_{1 \leq i \leq 3} \sum_{1 \leq j \leq 3} (\vec{p}_i^T \vec{p}_j I - \vec{p}_i \otimes \vec{p}_j) \frac{\text{Area}}{12} (1 + \delta_{ij})
 \end{aligned}$$



$$\vec{x} = L_1\vec{p}_1 + L_2\vec{p}_2 + L_3\vec{p}_3$$

Integration Rule for Tetrahedron

$$\int_{\vec{x} \in \text{Tet}} L_1^i L_2^j L_3^k L_4^l d\vec{x} = \frac{3! i! j! k! l! \text{Volume}}{(3 + i + j + k + l)!}$$



$$\vec{f}(\vec{x}) = L_1^i L_2^j L_3^k L_4^l$$

Baldoni, Velleda, Nicole Berline, Jesus De Loera, Matthias Köppe, and Michèle Vergne. "How to integrate a polynomial over a simplex." *Mathematics of Computation* 80, no. 273 (2011): 297–325. <https://arxiv.org/abs/0809.2083>