

Einstein's Summation Rule

- Repeated indices are summed over

I have made a great discovery in mathematics!



inner product

$$a \cdot b = a_i b_i = \sum_i a_i b_i$$

Frobenius inner product

$$\langle A, B \rangle_F = A_{ij} B_{ij} = \sum_j \sum_i A_{ij} B_{ij}$$

Let's Practice Einstein's Summation Rule

$$\text{tr}(A) = ?$$

$$(A^T B)_{ij} = ?$$

$$\text{tr}(A^T B) = ?$$

$$(\vec{a}^T \vec{b} I)_{ij} = ?$$

Frobenius Inner Product $\langle A, B \rangle_F = A_{ij}B_{ij}$

$$A = [\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n]$$

$$B = [\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n]$$

$$\langle A, B \rangle_F = \sum_i \vec{a}_i^T \vec{b}_i$$

$$\langle RA, B \rangle_F = R_{ik}A_{kj}B_{ij} = \sum_i (R\vec{a}_i)^T \vec{b}_i$$

$$= A_{kj}(R^T B)_{kj} = \langle A, R^T B \rangle_F = \sum_i \vec{a}_i^T R\vec{b}_i$$

$$= R_{ik}(BA^T)_{ik} = \langle R, BA^T \rangle_F = \left\langle R, \sum_i \vec{b}_i \otimes \vec{a}_i \right\rangle_F$$

Tensor

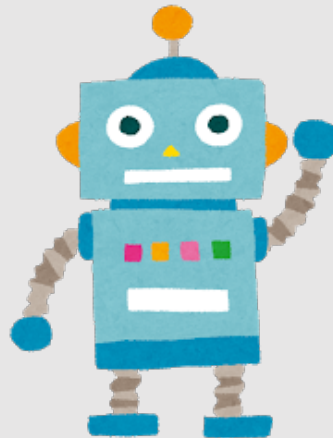
What is Tensor?



In mathematics, a tensor is an algebraic object that **describes a (multilinear) relationship** between sets of algebraic objects related to a vector space.

<https://en.wikipedia.org/wiki/Tensor>

scalar a
vector $\vec{u}$
tensor A



tensor



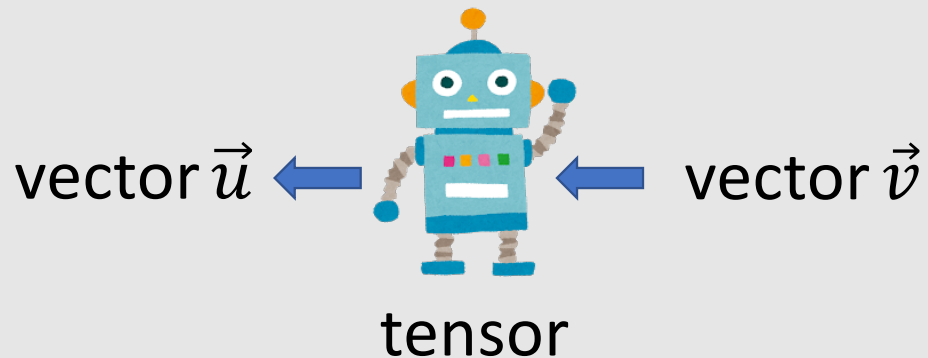
scalar b
vector $\vec{v}$
tensor B

Two ways to Understand 2nd-order Tensor

- Transformation by a tensor is give by the inner product

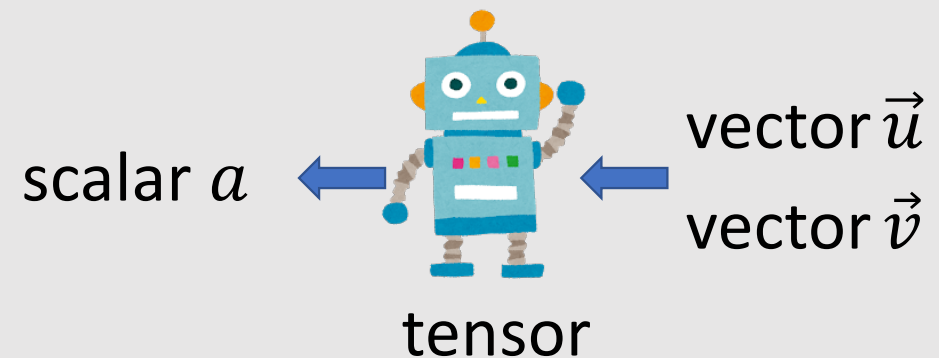
Linear form

$$\vec{u} = A \cdot \vec{v}$$



Quadratic form

$$a = \vec{u} \cdot (A \cdot \vec{v})$$



Outer Product (Tensor Product)

- Outer product makes a tensor from two vectors

$$\vec{a} \otimes \vec{b} \quad \longrightarrow \quad (\vec{a} \otimes \vec{b}) \cdot \vec{u} = \vec{a}(\vec{b} \cdot \vec{u})$$

- Tensor product $\vec{e} \otimes \vec{e}$ ($\|e\| = 1$) defines **projection**

Definition

Projection $\mathcal{P}$

$$\mathcal{P}(\mathcal{P}(x)) = \mathcal{P}$$

check it out!



Outer Product (Tensor Product)

- Transformation for vectors in the outer product

check it out!

$$(A\vec{a}) \otimes (B\vec{b}) = ?$$



Definition

$$\vec{a} \otimes \vec{b}$$

$$(\vec{a} \otimes \vec{b}) \cdot \vec{u} = \vec{a}(\vec{b} \cdot \vec{u})$$

Tensor + Basis = Matrix

- Inner product with a basis vector gives a coefficient
 - This is true even if the basis is not orthonormal

$$v_i = \vec{v} \cdot \vec{e}_i$$

$$a_{ij} = \vec{e}_i \cdot (A \cdot \vec{e}_j)$$

Tensor & Matrix: Common Misunderstanding



Wrong idea! Correct your thought!

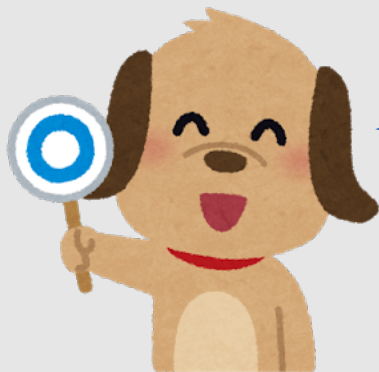
Tensor

=

Basis

×

Matrix
(coefficients)



Nice ! Go ahead!

Tensor

×

Basis

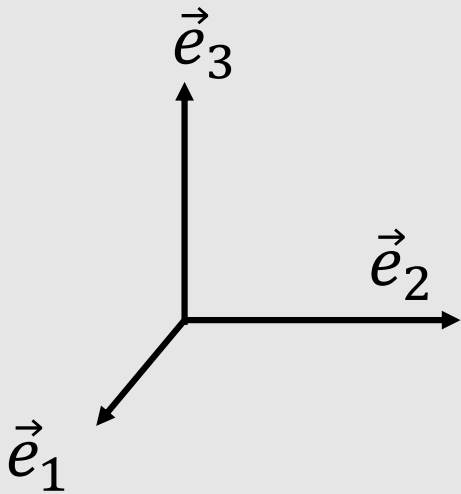
=

Matrix
(coefficients)

Orthonormal Coordinates

- Tensor can be written with bases and coefficients

$$e_i \cdot e_j = \delta_{ij}$$



$$v_i = \vec{v} \cdot \vec{e}_i$$

$$\rightarrow \vec{v} = v_i \vec{e}_i$$

$$a_{ij} = \vec{e}_i \cdot (A \cdot \vec{e}_j)$$

$$\rightarrow A = a_{ij} \vec{e}_i \otimes \vec{e}_j$$

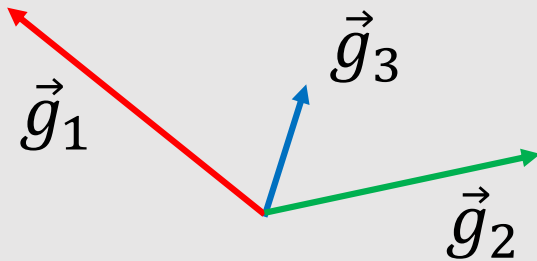


check it out!

Curvilinear Coordinates

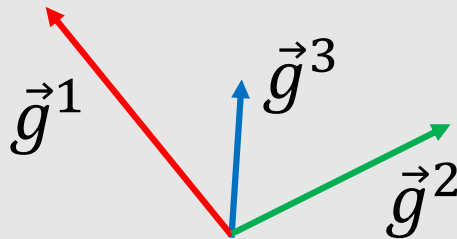
- Non-orthogonal and un-normalized bases
- Dual bases solve the problem

$$\vec{g}_i \cdot \vec{g}_j \neq \delta_{ij}$$



dual basis

$$\vec{g}_i \cdot \vec{g}^j = \delta_i^j$$



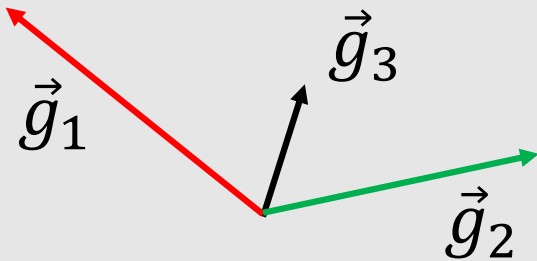
$$\vec{g}^1 = \frac{\vec{g}_2 \times \vec{g}_3}{\vec{g}_2 \cdot (\vec{g}_2 \times \vec{g}_3)}$$

$$\vec{g}^2 = \frac{\vec{g}_3 \times \vec{g}_1}{\vec{g}_3 \cdot (\vec{g}_3 \times \vec{g}_1)}$$

$$\vec{g}^3 = \frac{\vec{g}_1 \times \vec{g}_2}{\vec{g}_1 \cdot (\vec{g}_1 \times \vec{g}_2)}$$

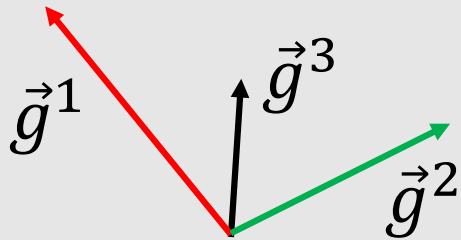
Curvilinear Coordinates

- Expression of a vector in curvilinear coordinates



dual basis

$$g_i \cdot g^j = \delta_i^j$$

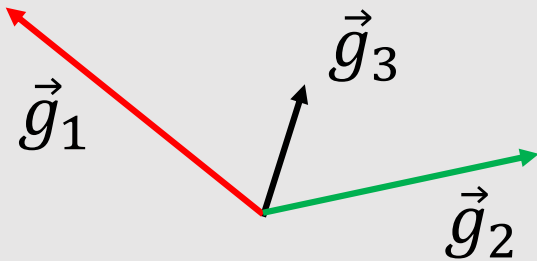


$$v^i = \vec{v} \cdot \vec{g}^i \quad \Rightarrow \quad \vec{v} = v^i \vec{g}_i$$

$$v_i = \vec{v} \cdot \vec{g}_i \quad \Rightarrow \quad \vec{v} = v_i \vec{g}^i$$

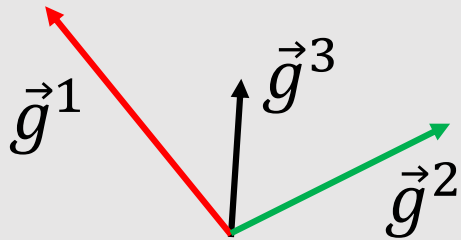
Curvilinear Coordinates

- Expression of a tensor in curvilinear coordinates



dual basis

$$g_i \cdot g^j = \delta_i^j$$



$$a^{ij} = \vec{g}^i \cdot (A \cdot \vec{g}^j) \Rightarrow A = a^{ij} \vec{g}_i \otimes \vec{g}_j$$

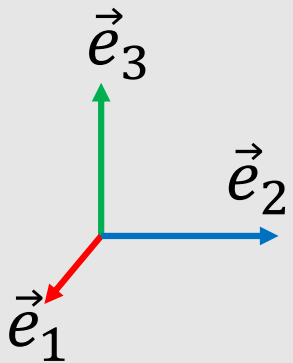
$$a_{ij} = \vec{g}_i \cdot (A \cdot \vec{g}_j) \Rightarrow A = a_{ij} \vec{g}^i \otimes \vec{g}^j$$

$$a^i_j = \vec{g}^i \cdot (A \cdot \vec{g}_i) \Rightarrow A = a^i_j \vec{g}_i \otimes \vec{g}^j$$

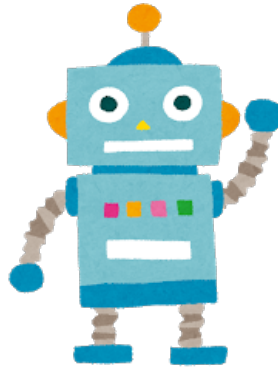
$$a_i^j = \vec{g}_i \cdot (A \cdot \vec{g}^j) \Rightarrow A = a_i^j \vec{g}^i \otimes \vec{g}_j$$

Coordinate Transformation

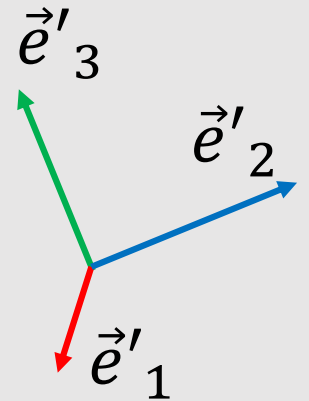
$$a_{ij} = \vec{e}_i \cdot (A \cdot \vec{e}_j)$$



tensor: A



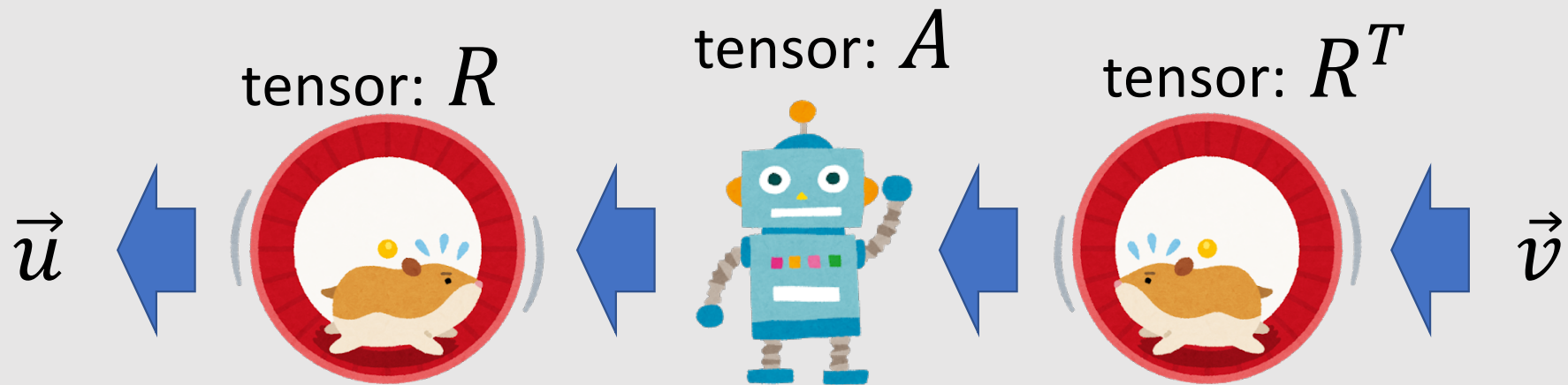
$$a'_{ij} = \vec{e}'_i \cdot (A \cdot \vec{e}'_j)$$



same tensor, different
(coefficient) matrix!

Rotation of a Tensor

- Un-rotating input and rotating output



$$\vec{u} = R A R^T \vec{v}$$

Rotation of a Tensor

- Rotating bases while using the same coefficients

$$A = a_{ij} \vec{e}_i \otimes \vec{e}_j$$

rotation of basis

$$\begin{aligned} A' &= a_{ij} (R\vec{e}_i) \otimes (R\vec{e}_j) \\ &= a_{ij} R(\vec{e}_i \otimes \vec{e}_j) R^T \\ &= R(a_{ij} \vec{e}_i \otimes \vec{e}_j) R^T \\ &= RAR^T \end{aligned}$$



check it out!

Simple Elastic Potential Energy for Continuum

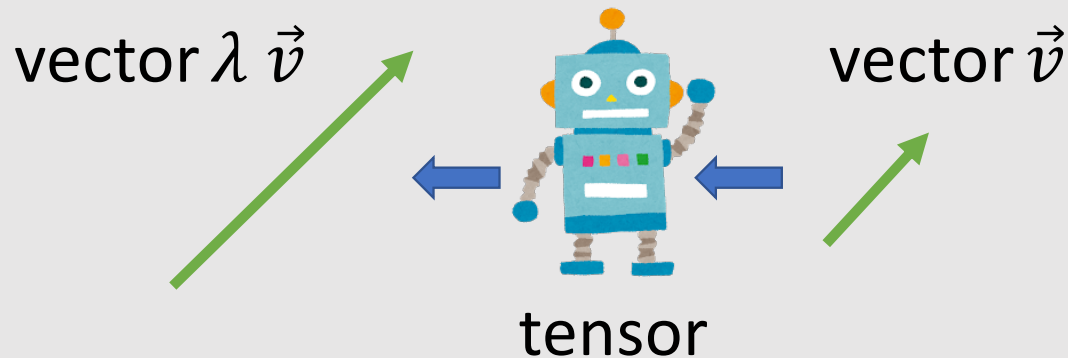


Eigenvalue of Symmetric Tensor

- Eigenvalue of tensor is defined without matrix & coordinate

Linear form

$$\lambda \vec{v} = A \cdot \vec{v}$$



Eigenvalues and Frobenius Norm

from characteristic equation

$$\text{tr}(A) = \lambda_1 + \lambda_2 + \lambda_3$$



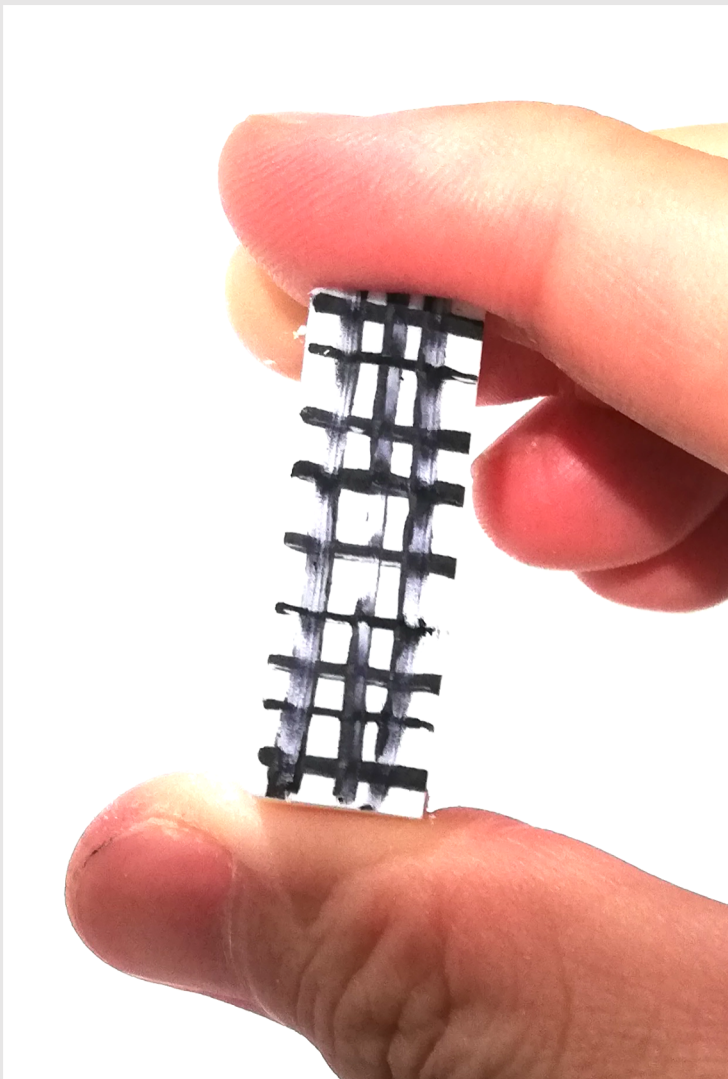
$$A^2 \vec{e} = \lambda^2 \vec{e}$$

$$\text{tr}(A^2) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$$

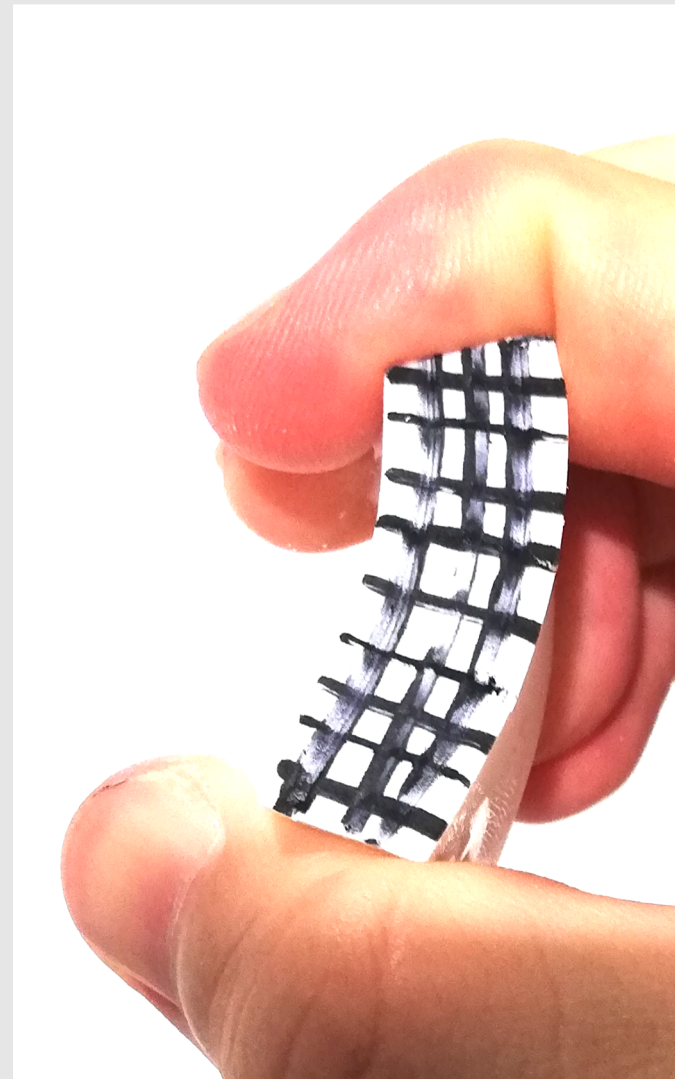
$$= \text{tr}(A^T A) = \sum_{1 \leq i, j \leq 3} a_{ij}^2 = \|A\|_F^2$$

Frobenius norm

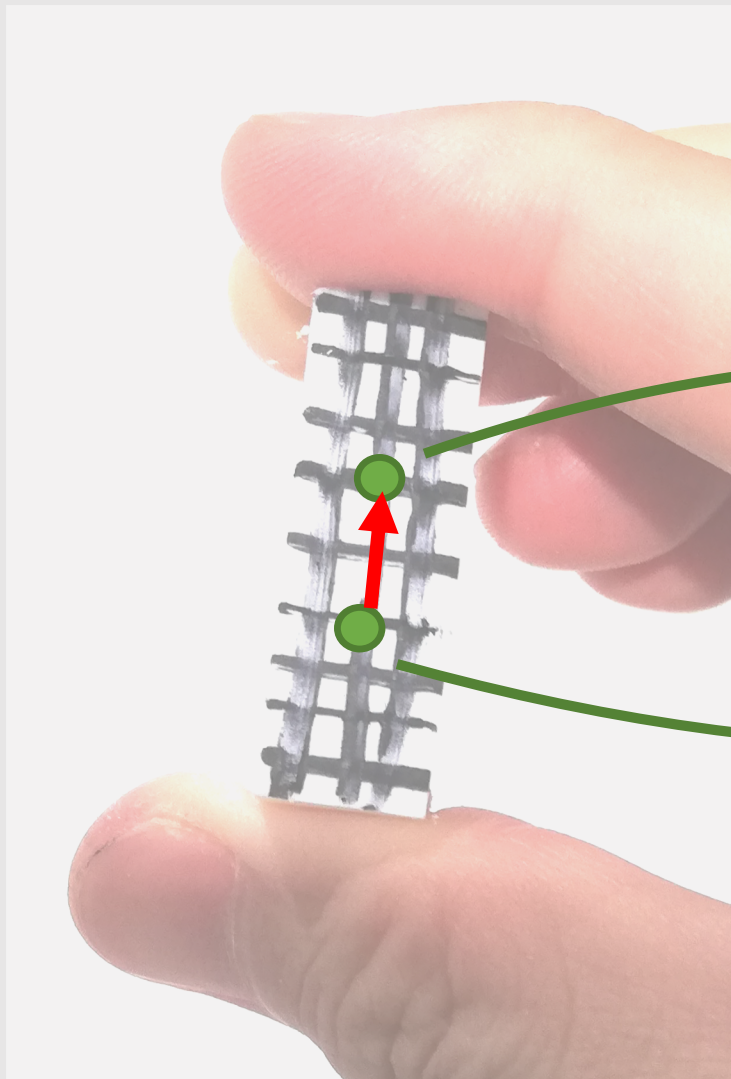
rest shape



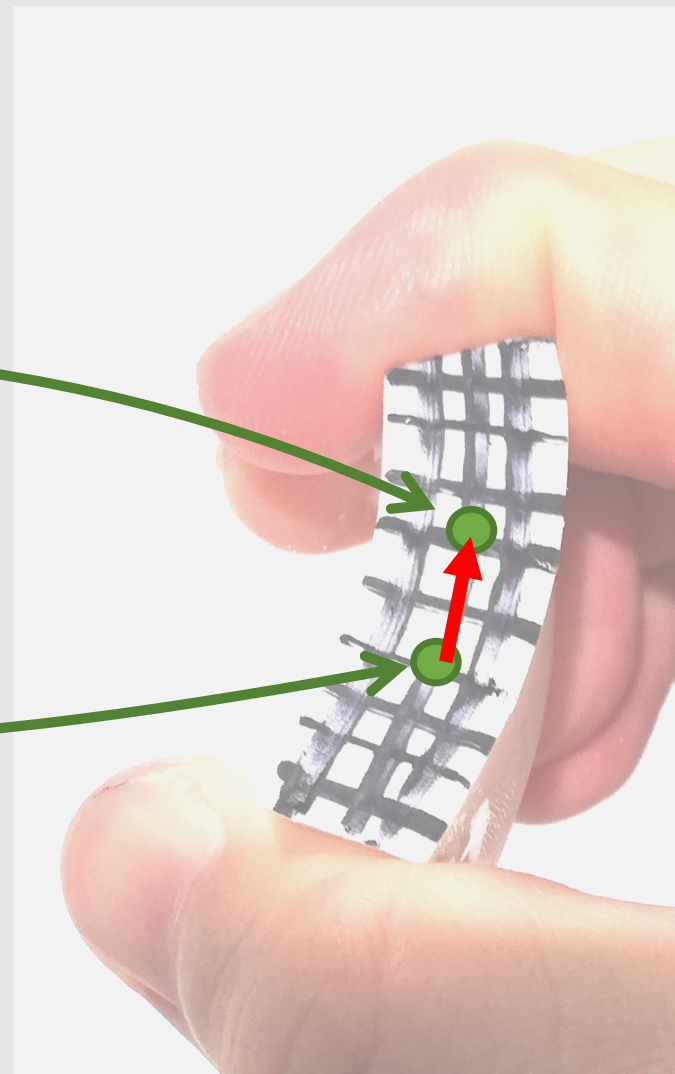
deformed shape



rest shape

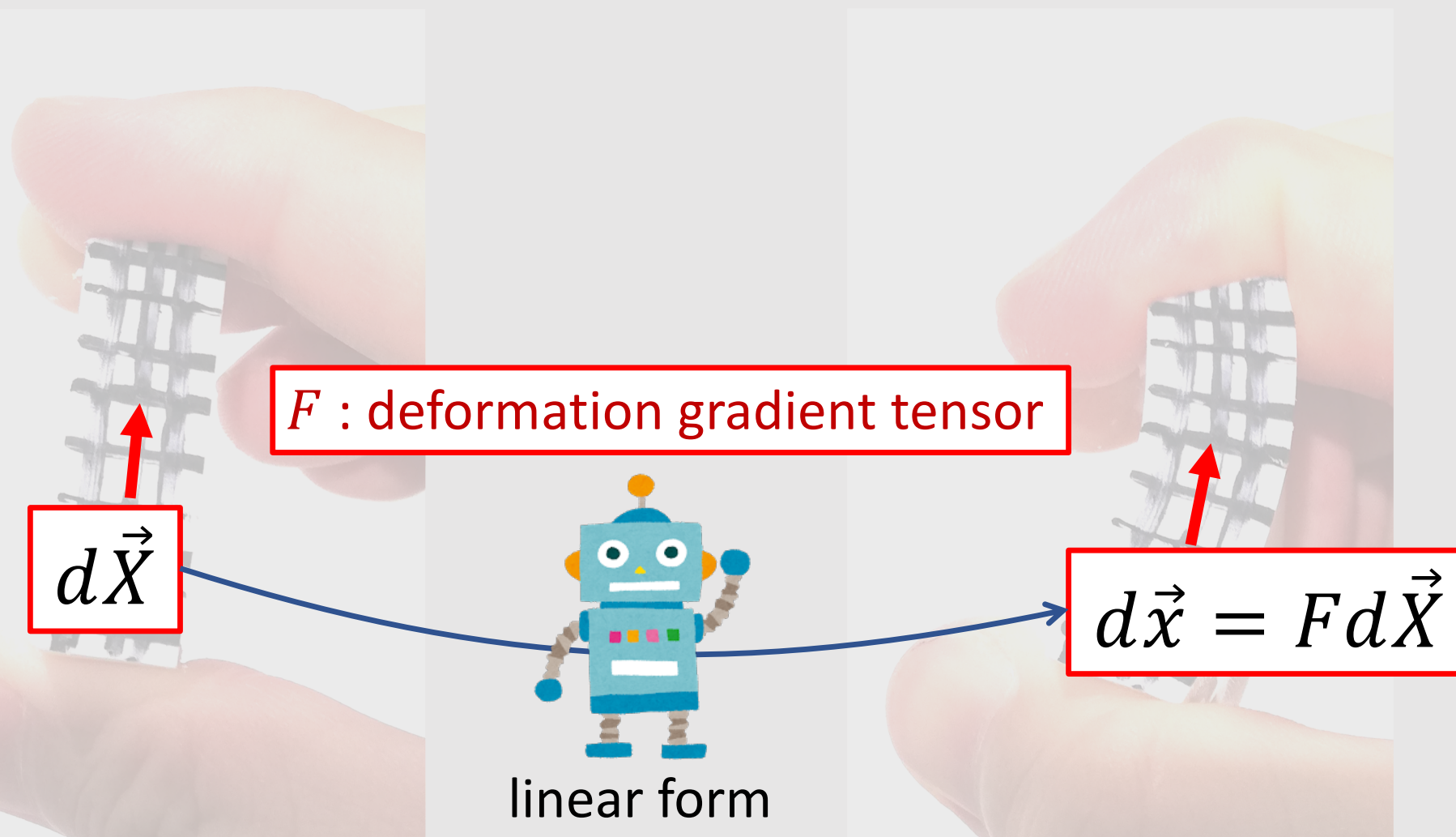


deformed shape

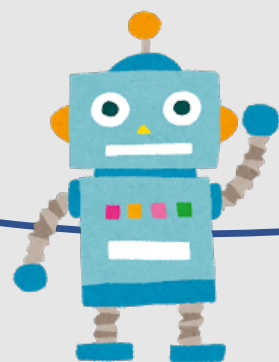


rest shape

deformed shape


$$d\vec{X}$$

F : deformation gradient tensor

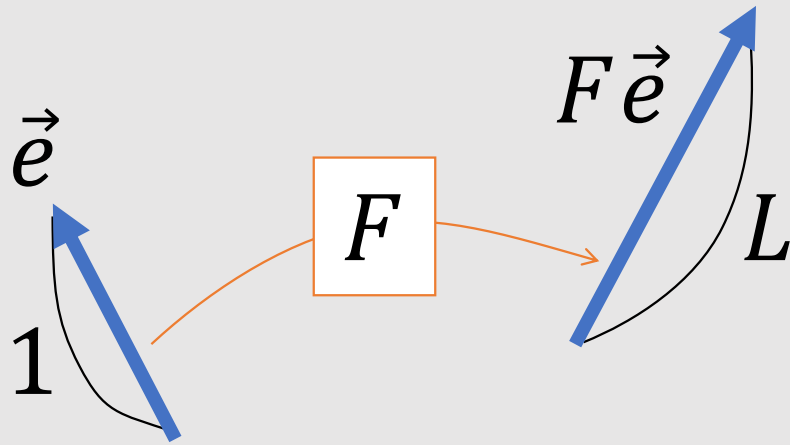


linear form

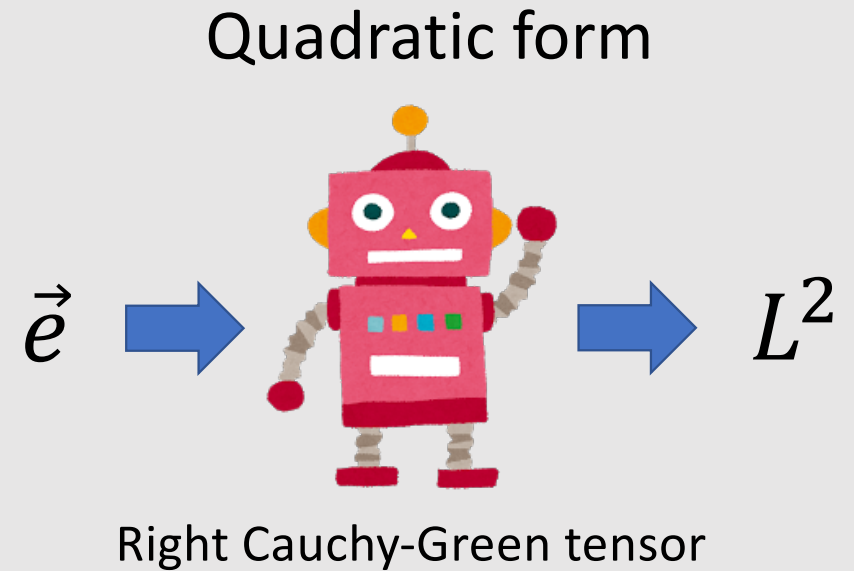
$$d\vec{x} = F d\vec{X}$$

Gram Matrix $F^T F$ Stands for Length Change

- $C = F^T F$: right Cauchy-Green tensor

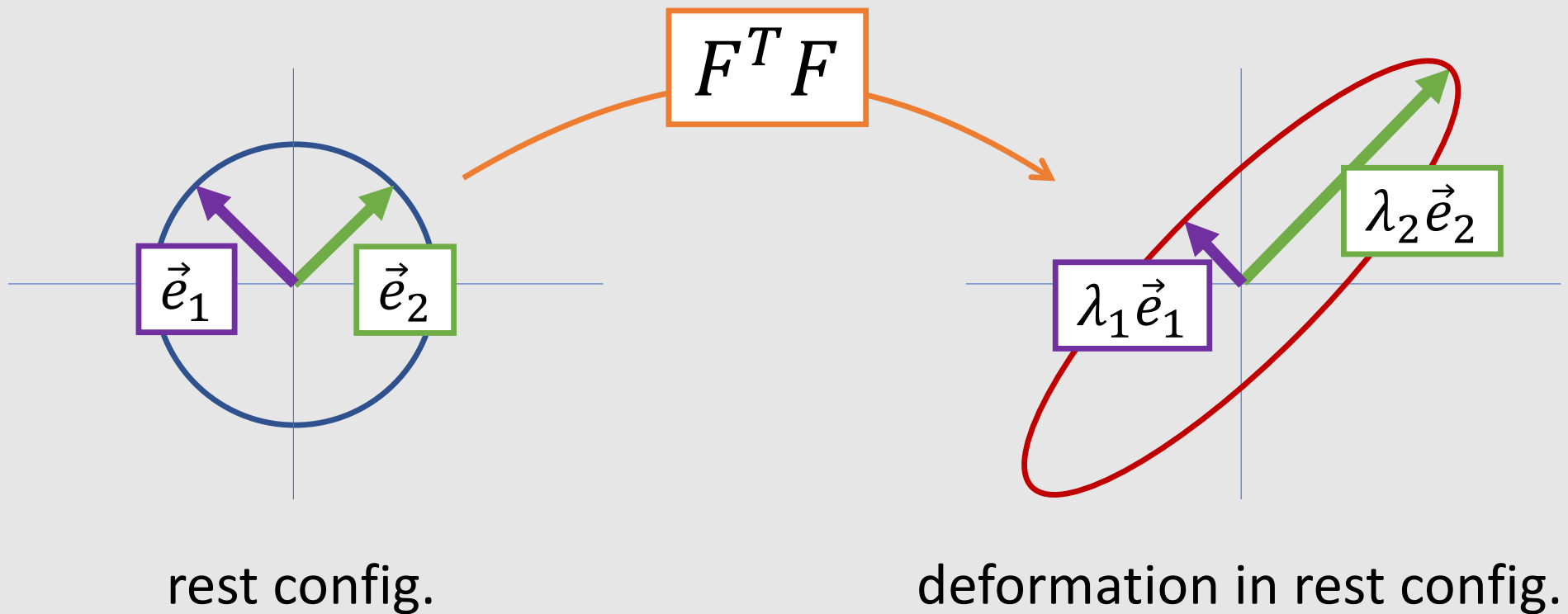


$$L^2(\vec{e}) = \vec{e}^T F^T F \vec{e}$$

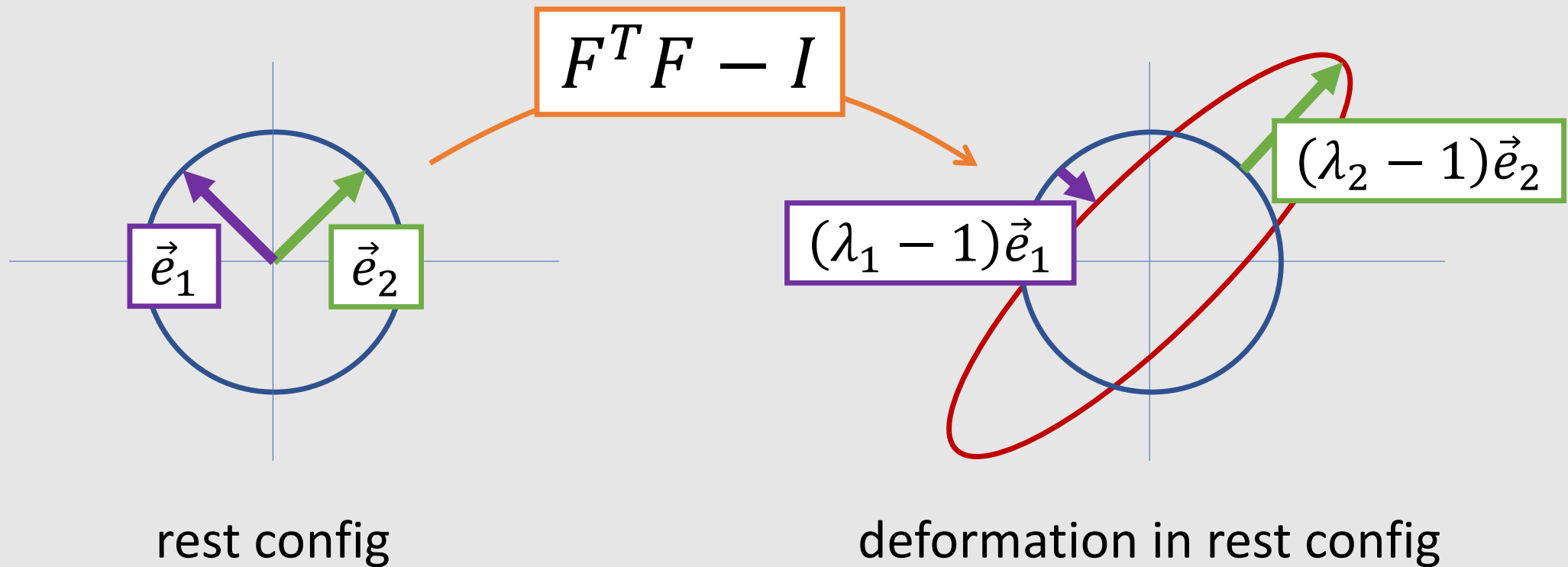


Eigenvalue: Right Cauchy Green Tensor $F^T F$

- eigenvalues: $\lambda_1, \lambda_2, \lambda_3$



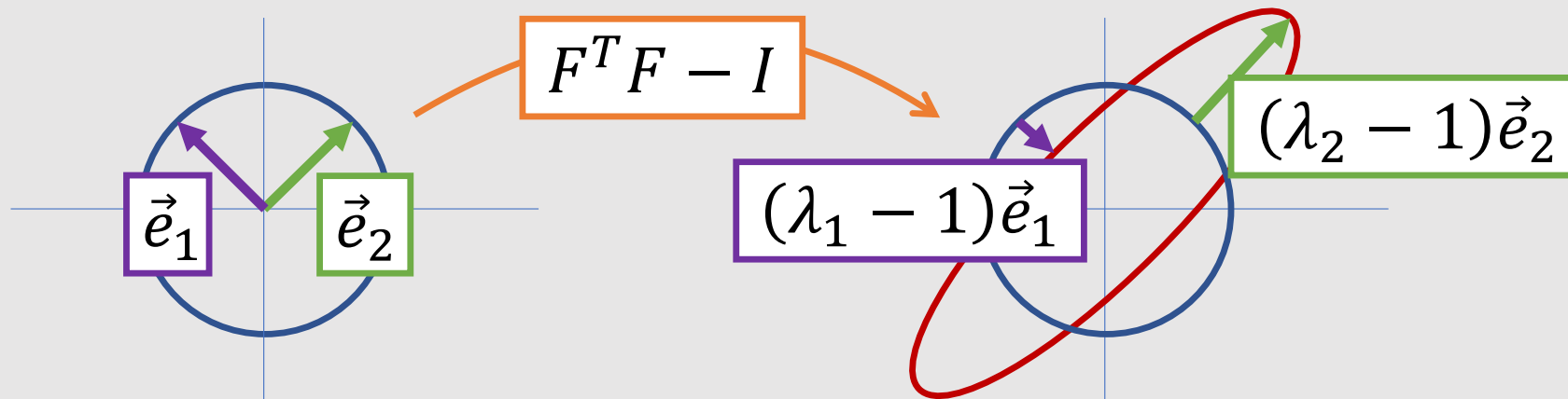
Eigenvalue: Green Lagrange Tensor $F^T F - I$



Making Energy from Eigenvalue

- Energy for isotropic material

$$W(F) = \|F^T F - I\|_F^2 = (\lambda_1 - 1)^2 + (\lambda_2 - 1)^2 + (\lambda_3 - 1)^2$$



Let's put penalty on the absolute value of $(\lambda_1 - 1)$, $(\lambda_2 - 1)$ and $(\lambda_3 - 1)$



Finite Element Method

What is Finite Element Method?

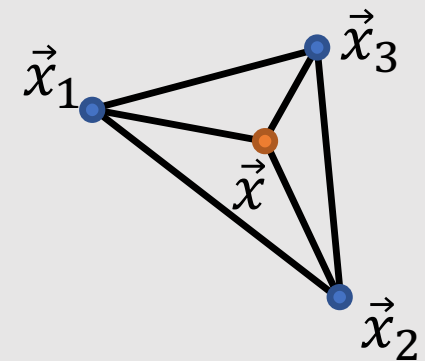
- Solution by energy minimization

$$\vec{x}_{solution} = \operatorname{argmin}_{\vec{x}} W(\vec{x})$$



- Value inside element is interpolated

$$\vec{x} = \sum_{i \in Nodes} w_i \vec{x}_i$$

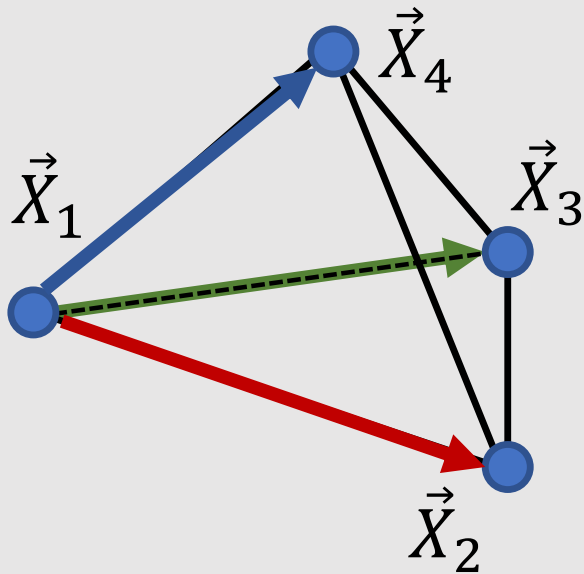


- Energy is integrated inside element and summed

$$W(\vec{x}) = \sum_{e \in Elements} W_e(\vec{x})$$

Deformation Gradient Tensor F for Tet.

rest shape

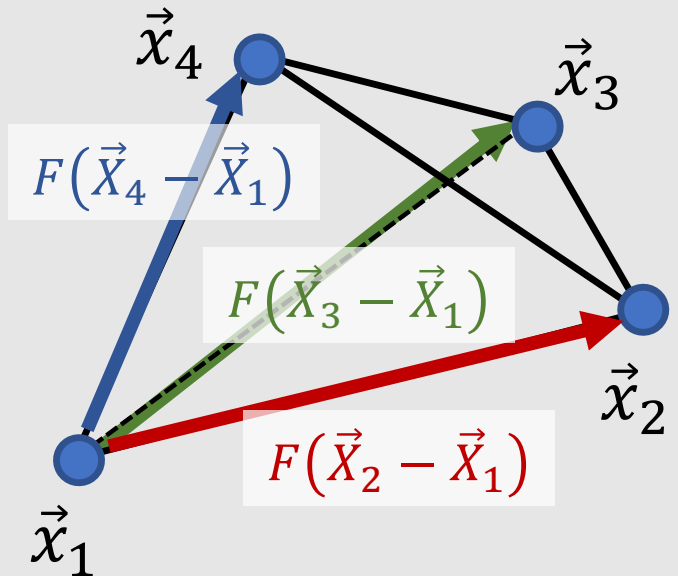


$$(\vec{x}_2 - \vec{x}_1) = F(\vec{X}_2 - \vec{X}_1)$$

$$(\vec{x}_3 - \vec{x}_1) = F(\vec{X}_3 - \vec{X}_1)$$

$$(\vec{x}_4 - \vec{x}_1) = F(\vec{X}_4 - \vec{X}_1)$$

deformed shape

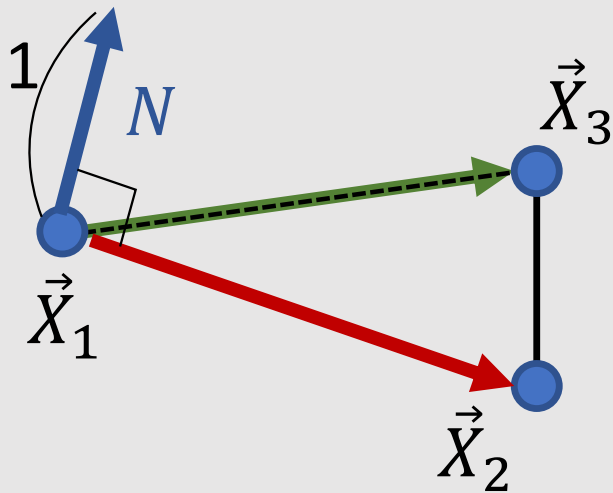


$$[\vec{x}_2 - \vec{x}_1, \vec{x}_3 - \vec{x}_1, \vec{x}_4 - \vec{x}_1] = F[\vec{X}_2 - \vec{X}_1, \vec{X}_3 - \vec{X}_1, \vec{X}_4 - \vec{X}_1]$$

$$F = [\vec{x}_2 - \vec{x}_1, \vec{x}_3 - \vec{x}_1, \vec{x}_4 - \vec{x}_1][\vec{X}_2 - \vec{X}_1, \vec{X}_3 - \vec{X}_1, \vec{X}_4 - \vec{X}_1]^{-1}$$

Deformation Gradient Tensor F for 3D Tri.

rest shape

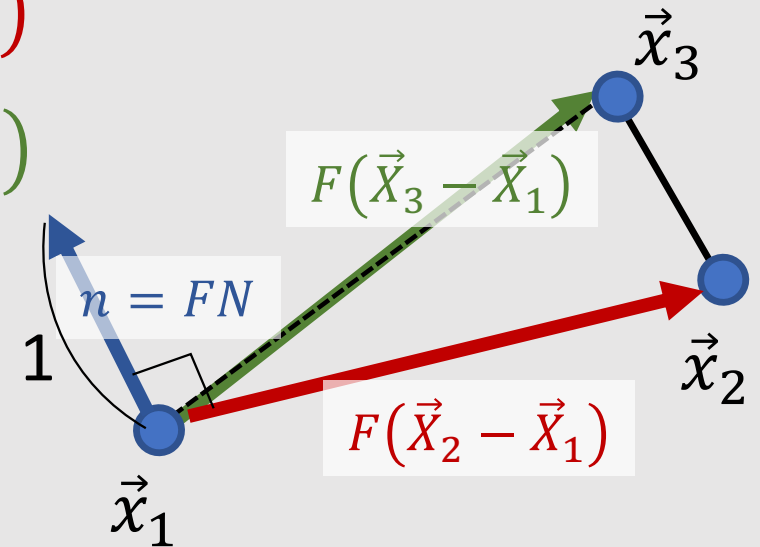


$$(\vec{x}_2 - \vec{x}_1) = F(\vec{X}_2 - \vec{X}_1)$$

$$(\vec{x}_3 - \vec{x}_1) = F(\vec{X}_3 - \vec{X}_1)$$

$$n = FN$$

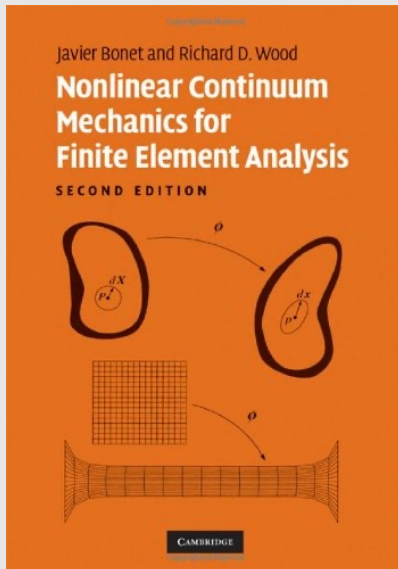
deformed shape



$$[\vec{x}_2 - \vec{x}_1, \vec{x}_3 - \vec{x}_1, n] = F[\vec{X}_2 - \vec{X}_1, \vec{X}_3 - \vec{X}_1, N]$$

$$F = [\vec{x}_2 - \vec{x}_1, \vec{x}_3 - \vec{x}_1, n][\vec{X}_2 - \vec{X}_1, \vec{X}_3 - \vec{X}_1, N]^{-1}$$

Reference



- Bonet, Javier, and Richard D. Wood. 1997. *Nonlinear continuum mechanics for finite element analysis*. Cambridge: Cambridge University Press.

Linearization of Green-Lagrange Strain

Green Lagrange Strain: $E = F^T F - I$

More Realistic Elasticity Model

Green Lagrange Strain: $E = F^T F - I$

$$W(F) = \|F^T F - I\|_F^2$$

$$W(F) = \|F^T F - I\|_F^2$$

4th-order Tensor

Definition of Inner product in higher dimension

